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Initial boundary value problems for a quantum hydrodynamic model of semiconductors: Asymptotic behaviors and classical limits

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Abstract

The present paper proves the existence and the asymptotic stability of a stationary solution to the initial boundary value problem for a quantum hydrodynamic model of semiconductors over a one-dimensional bounded domain. We also discuss on a singular limit from this model to a classical hydrodynamic model without quantum effects. Precisely, we prove that a solution for the quantum model converges to that for the hydrodynamic model as the Planck constant tends to zero. Here we adopt a non-linear boundary condition which means quantum effect vanishes on the boundary. In the previous researches, the existence and the asymptotic stability of a stationary solution are proved under the assumption that a doping profile is flat, which makes the stationary solution also flat. However, the typical doping profile in actual devices does not satisfy this assumption. Thus, we prove the above theorems without this flatness assumption. Firstly, the existence of the stationary solution is proved by the Leray–Schauder fixed-point theorem. Secondly, we show the asymptotic stability theorem by using an elementary energy method, where the equation for an energy form plays an essential role. Finally, the classical limit is considered by using the energy method again.

Keywords: Quantum Euler–Poisson equations; Stationary wave; Large-time behavior; Singular limit; Asymptotic stability

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1. Introduction

The aim of the present paper is to consider the existence and the asymptotic stability of a stationary solution to the initial boundary value problem for a one-dimensional quantum hydrodynamic model of semiconductors. We also study a singular limit of the solution. The quantum hydrodynamic model of semiconductors is given by the system of equations, which contains a momentum relaxation term taking collisions with atoms in the semiconductor crystal into account, and are coupled with the Poisson equation,

$$\rho_t + j_x = 0, \tag{1.1a}$$

$$j_t + \left(\frac{j^2}{\rho} + p(\rho)\right)_x - \varepsilon^2 \rho \left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x = \rho\phi_x - j, \tag{1.1b}$$

$$\phi_{xx} = \rho - D, \tag{1.1c}$$

where the quantum effect is concentrated on a dispersion term based on the quantum (Bohm) potential. The unknown functions ρ, j and ϕ stand for the electron density, the electric current and the electrostatic potential, respectively. The positive constant ε is called the scaled Planck constant, which is equivalent to the Planck constant $\hbar$. Since we study the isothermal flow, the pressure p is a function of the electron density ρ given by

$$p = p(\rho) = K\rho, \tag{1.2}$$

where K is the positive constant. A doping profile $D \in \mathcal{B}^0(\overline{\Omega})$, which determines the electric property of semiconductor devices, is a function of the spatial variable $x \in \overline{\Omega} := [0, 1]$ and satisfies

$$\inf_{x \in \overline{\Omega}} D(x) > 0. \tag{1.3}$$

The system (1.1) is derived from the Wigner–Boltzmann equation through the moment expansion under the assumption $\varepsilon \ll 1$ (see [1,4] in details). We study (1.1) over the bounded domain $\Omega := (0, 1)$ and obtain the asymptotic behavior of a solution (ρ, j, ϕ) to (1.1). Moreover, its singular limit, $\varepsilon \to 0$ in (1.1), is also considered. The latter problem is called a classical limit.

The initial and the boundary conditions to the system (1.1) are prescribed as

$$(\rho, j)(0, x) = (\rho_0, j_0)(x), \tag{1.4}$$

$$\rho(t, 0) = \rho_l > 0, \qquad \rho(t, 1) = \rho_r > 0, \tag{1.5}$$

$$(\sqrt{\rho})_{xx}(t, 0) = (\sqrt{\rho})_{xx}(t, 1) = 0, \tag{1.6}$$

$$\phi(t, 0) = 0, \qquad \phi(t, 1) = \phi_r > 0, \tag{1.7}$$

where ρ_l, ρ_r and ϕ_r are given constants. It is also assumed that the initial data (1.4) is compatible with the boundary data (1.5)–(1.7) in order to establish the existence of a classical solution. Namely, it is assumed that

$$\rho_0(0)=\rho_l, \qquad \rho_0(1)=\rho_r, \qquad j_{0x}(0)=j_{0x}(1)=0,$$
$$(\sqrt{\rho_0})_{xx}(0)=(\sqrt{\rho_0})_{xx}(1)=0. \tag{1.8}$$

Integrating (1.1c) with the aid of the boundary condition (1.7) yields an explicit formula of the electrostatic potential

$$\begin{aligned}\phi(t,x) &= \Phi[\rho](t,x)\\ &:= \int_0^x\int_0^y (\rho-D)(t,z)\,dz\,dy + \left(\phi_r - \int_0^1\int_0^y (\rho-D)(t,z)\,dz\,dy\right)x.\end{aligned} \tag{1.9}$$

In consideration of the existence of the solution to (1.1), the following properties play essential roles:

$$\inf_{x\in\Omega} S[\rho,j]>0, \qquad S[\rho,j]:=p'(\rho)-\frac{j^2}{\rho^2}, \tag{1.10a}$$
$$\inf_{x\in\Omega} \rho>0. \tag{1.10b}$$

The condition (1.10a) is called a subsonic condition, (1.10b) is a positivity of the density, respectively. In Section 3, we construct the solution (ρ, j, ϕ) in the region where the conditions (1.10) hold under the assumption that the initial data satisfies the same assumptions

$$\inf_{x\in\Omega} S[\rho_0,j_0]>0, \qquad \inf_{x\in\Omega} \rho_0(x)>0. \tag{1.11}$$

Asymptotic stability of stationary solution. To consider the initial boundary value problem (1.1) and (1.4)–(1.7), it is convenient to rewrite the problem (1.1) and (1.4)–(1.7) for (ω, j, ϕ), where $\omega := \sqrt{\rho}$, as

$$2\omega\omega_t + j_x = 0, \tag{1.12a}$$
$$j_t + 2S\big[\omega^2, j\big]\omega\omega_x + 2\frac{j}{\omega^2}j_x - \varepsilon^2\omega^2\left(\frac{\omega_{xx}}{\omega}\right)_x = \omega^2\phi_x - j, \tag{1.12b}$$
$$\phi_{xx} = \omega^2 - D \tag{1.12c}$$

with the initial and the boundary data

$$(\omega, j)(0,x) = (\omega_0, j_0)(x) := (\sqrt{\rho_0}, j_0)(x), \tag{1.13}$$
$$\omega(t,0)=\omega_l := \sqrt{\rho_l}>0, \qquad \omega(t,1)=\omega_r := \sqrt{\rho_r}>0, \tag{1.14}$$
$$\omega_{xx}(t,0)=\omega_{xx}(t,1)=0, \tag{1.15}$$
$$\phi(t,0)=0, \qquad \phi(t,1)=\phi_r. \tag{1.16}$$

Apparently, (1.1) is equivalent to (1.12), if the density ρ is positive. Thus once we prove the existence of a solution to the initial boundary value problem (1.12)–(1.16) for (ω, j, ϕ) with

$\omega > 0$, the existence of the solution to the original problem (1.1) and (1.4)–(1.7) immediately follows.

We will see that the asymptotic behavior of solution (ρ, j, ϕ) to the problem (1.1) is a stationary solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$, which is a solution to (1.1) independent of a time variable t, satisfying the boundary conditions (1.5)–(1.7). Hence, it verifies a system of equations

$$\tilde{j}_x = 0, \tag{1.17a}$$

$$S[\tilde{\rho}, \tilde{j}]\tilde{\rho}_x - \varepsilon^2\tilde{\rho}\left(\frac{(\sqrt{\tilde{\rho}})_{xx}}{\sqrt{\tilde{\rho}}}\right)_x = \tilde{\rho}\tilde{\phi}_x - \tilde{j}, \tag{1.17b}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - D \tag{1.17c}$$

and boundary conditions

$$\tilde{\rho}(0) = \rho_l > 0, \qquad \tilde{\rho}(1) = \rho_r > 0, \tag{1.18}$$

$$(\sqrt{\tilde{\rho}})_{xx}(0) = (\sqrt{\tilde{\rho}})_{xx}(1) = 0, \tag{1.19}$$

$$\tilde{\phi}(0) = 0, \qquad \tilde{\phi}(1) = \phi_r > 0. \tag{1.20}$$

Letting $\tilde{\omega} := \sqrt{\tilde{\rho}}$, we have from (1.17) and (1.18)–(1.20) that $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ satisfies a system of equations

$$2S\big[\tilde{\omega}^2, \tilde{j}\big]\tilde{\omega}\tilde{\omega}_x - \varepsilon^2\tilde{\omega}^2\left(\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)_x = \tilde{\omega}^2\tilde{\phi}_x - \tilde{j}, \tag{1.21a}$$

$$\tilde{\phi}_{xx} = \tilde{\omega}^2 - D \tag{1.21b}$$

and (1.17a) with boundary conditions

$$\tilde{\omega}(0) = \omega_l > 0, \qquad \tilde{\omega}(1) = \omega_r > 0, \tag{1.22}$$

$$\tilde{\omega}_{xx}(0) = \tilde{\omega}_{xx}(1) = 0 \tag{1.23}$$

and (1.20). Divide (1.21a) by $\tilde{\omega}^2$ and integrate the resultant equation over $(0, x)$, and then use the boundary conditions (1.20), (1.22) and (1.23). Moreover, apply the Green formula to Eq. (1.21b) together with the boundary condition (1.20). These procedures yield that

$$\varepsilon^2\frac{\tilde{\omega}_{xx}}{\tilde{\omega}} = F\big(\tilde{\omega}^2, \tilde{j}\big) - F(\rho_l, \tilde{j}) - \tilde{\phi} + \int_0^x \frac{\tilde{j}}{\tilde{\omega}^2}(y)\,dy, \tag{1.24a}$$

$$\tilde{\phi} = \mathcal{G}\big[\tilde{\omega}^2\big] := \int_0^1 G(x,\xi)\big(\tilde{\omega}^2 - D\big)(\xi)\,d\xi + \phi_r x, \tag{1.24b}$$

$$F(\xi, \zeta) := \frac{\zeta^2}{2\xi^2} + K\log\xi, \qquad G(x,\xi) := \begin{cases} x(\xi - 1) & \text{for } x < \xi, \\ \xi(x - 1) & \text{for } x > \xi. \end{cases} \tag{1.24c}$$

Substituting $x = 1$ in (1.24a), we have from (1.22) and (1.23) the current-voltage relationship

$$\phi_r = F(\rho_r, \tilde{j}) - F(\rho_l, \tilde{j}) + \tilde{j} \int_0^1 \frac{1}{\tilde{\rho}}\, dx. \tag{1.25}$$

In the proofs of following theorems, especially in the discussions on the existence and the asymptotic stability of the stationary solution, the strength of the boundary data

$$\delta := |\rho_r - \rho_l| + |\phi_r| \tag{1.26}$$

plays a crucial role. We are now at the position to state one of the main results in the present paper. The existence of the stationary solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ is summarized in the next lemma.

Lemma 1.1. *Let the doping profile and the boundary data satisfy conditions* (1.3)*,* (1.5) *and* (1.7)*. For an arbitrary ρ_l, there exist positive constants δ_1 and ε_1 such that if $\delta \leqslant \delta_1$ and $\varepsilon \leqslant \varepsilon_1$, then the stationary problem* (1.17)–(1.20) *has a unique solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi}) \in \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ satisfying the conditions* (1.10)*.*

Proof. This lemma follows from Lemmas 2.1 and 2.4. □

The electric current $\tilde{j}$, in Lemma 1.1, is given by the formula

$$\tilde{j} = \mathcal{J}[\tilde{\rho}] := 2B_b \left\{ \int_0^1 \tilde{\rho}^{-1}\, dx + \sqrt{\left(\int_0^1 \tilde{\rho}^{-1}\, dx \right)^2 + 2B_b\big(\rho_r^{-2} - \rho_l^{-2}\big)} \right\}^{-1},$$
$$B_b := \phi_r - K\{\log \rho_r - \log \rho_l\}. \tag{1.27}$$

In order to construct the solution to the non-stationary problem (1.1) and (1.4)–(1.7), we employ the function space

$$\overline{\mathfrak{X}}_i^l\big([0,T]\big) := \bigcap_{k=0}^{[i/2]} C^k\big([0,T]; H^{l+i-2k}(\Omega)\big) \quad \text{for } i, l = 0, 1, 2, \ldots,$$
$$\overline{\mathfrak{X}}_i\big([0,T]\big) := \overline{\mathfrak{X}}_i^0\big([0,T]\big) \quad \text{for } i = 0, 1, 2, \ldots,$$
$$\mathfrak{Y} := C^2\big([0,T]; H^2(\Omega)\big),$$

where $[\mu]$ denotes the largest integer which is less than or equal to μ. The stability of the stationary solution is stated in the next theorem.

Theorem 1.2. *Let $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ be the stationary solution of* (1.17)–(1.20)*. Suppose that the initial data $(\rho_0, j_0) \in H^4(\Omega) \times H^3(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (1.5)*,* (1.7)*,* (1.8) *and* (1.11)*. Then there exists a positive constant δ_2 such that if $\delta + \varepsilon + \|(\rho_0 - \tilde{\rho}, j_0 - \tilde{j})\|_2 + \|(\varepsilon\partial_x^3\{\rho_0 - \tilde{\rho}\}, \varepsilon\partial_x^3\{j_0 - \tilde{j}\}, \varepsilon^2\partial_x^4\{\rho_0 - \tilde{\rho}\})\| \leqslant \delta_2$, the initial boundary value problem* (1.1) *and* (1.4)–(1.7) *has a unique solution (ρ, j, ϕ) in the space $\overline{\mathfrak{X}}_4([0,\infty)) \times \overline{\mathfrak{X}}_3([0,\infty)) \times \mathfrak{Y}([0,\infty))$.*

Moreover, the solution (ρ, j, ϕ) *verifies the additional regularity* $\phi - \tilde{\phi} \in \overline{\mathfrak{X}}_4^2([0,\infty))$ *and the decay estimate*

$$\begin{aligned}&\left\|(\rho-\tilde{\rho}, j-\tilde{j})(t)\right\|_2 + \left\|\left(\varepsilon\partial_x^3\{\rho-\tilde{\rho}\}, \varepsilon\partial_x^3\{j-\tilde{j}\}, \varepsilon^2\partial_x^4\{\rho-\tilde{\rho}\}\right)(t)\right\| + \left\|(\phi-\tilde{\phi})(t)\right\|_4 \\ &\quad \leqslant C\left(\left\|(\rho_0-\tilde{\rho}, j_0-\tilde{j})\right\|_2 + \left\|\left(\varepsilon\partial_x^3\{\rho_0-\tilde{\rho}\}, \varepsilon\partial_x^3\{j_0-\tilde{j}\}, \varepsilon^2\partial_x^4\{\rho_0-\tilde{\rho}\}\right)\right\|\right)e^{-\alpha_1 t}, \qquad (1.28)\end{aligned}$$

where C *and* α_1 *are positive constants, independent of* t *and* ε.

Remark 1.3. The similar result as the above theorem has been proved in [8] under the assumption that the doping profile is flat, that is, $|D(x) - \rho_l| \ll 1$. However this assumption is too narrow to cover actual devices (see [4]). This assumption also makes the problem easy since the stationary solution becomes flat, i.e. $|(\tilde{\rho}_x, \tilde{\phi}_x)|_0 \ll 1$. In the above theorem, we do not need this assumption. On the other hand, the assumption $\varepsilon \ll 1$ is admissible from the physical point of view since the system (1.1) is derived under this assumption.

Classical limit. The second purpose of the present research is to study the singular limit of the solution (ρ, j, ϕ) to the initial boundary value problem (1.1) and (1.4)–(1.7) as the parameter ε tends to zero. To this end, let (ρ^0, j^0, ϕ^0) be a solution to the hydrodynamic model, which is obtained by substituting $\varepsilon = 0$ in (1.1). On the other hand, we write solutions to (1.1) and (1.4)–(1.7) with the suffix ε as $(\rho^\varepsilon, j^\varepsilon, \phi^\varepsilon)$ for clarity without confusion. Then we have the hydrodynamic model satisfied by (ρ^0, j^0, ϕ^0)

$$\rho_t^0 + j_x^0 = 0, \qquad (1.29a)$$

$$j_t^0 + \left(\frac{(j^0)^2}{\rho^0} + p(\rho^0)\right)_x = \rho^0\phi_x^0 - j^0, \qquad (1.29b)$$

$$\phi_{xx}^0 = \rho^0 - D. \qquad (1.29c)$$

The initial and the boundary data to (1.29) are prescribed by (1.4), (1.5) and (1.7). In the research [5], the global solvability and the asymptotic stability of a stationary solution to (1.29) are proved in the region where the subsonic condition (1.10a) and the positivity of the density (1.10b) hold (see Lemmas 1.4 and 1.5 below). Since the stationary solution $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ to (1.29) is independent of time value t, it satisfies the equations

$$\tilde{j}_x^0 = 0, \qquad (1.30a)$$

$$S[\tilde{\rho}^0, \tilde{j}^0]\tilde{\rho}_x^0 = \tilde{\rho}^0\tilde{\phi}_x^0 - \tilde{j}^0, \qquad (1.30b)$$

$$\tilde{\phi}_{xx}^0 = \tilde{\rho}^0 - D \qquad (1.30c)$$

with the boundary conditions (1.18) and (1.20). The existence of the stationary solution to the hydrodynamic model is ensured in the next lemma (see [13]).

Lemma 1.4. *Let the doping profile and the boundary data satisfy conditions* (1.3), (1.5) *and* (1.7). *For an arbitrary* ρ_l, *there exists a positive constant* δ_3 *such that if* $\delta \leqslant \delta_3$, *then the stationary* *problem* (1.18), (1.20) *and* (1.30) *has a unique solution* $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)(x)$ *satisfying the conditions* (1.10) *in the space* $\mathcal{B}^2(\overline{\Omega})$. *Moreover the stationary solution satisfies the estimates*

$$0 < c \leqslant \tilde{\rho}^0 \leqslant C, \qquad |\tilde{j}^0|_0 \leqslant C\delta, \qquad |\tilde{\rho}^0|_2 + |\tilde{\phi}^0|_2 \leqslant C, \tag{1.31}$$

where c *and* C *are positive constants independent of* ρ_r *and* ϕ_r.

The stability of the stationary solution $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ is proved in [5] (also, see [13]).

Lemma 1.5. *Let* $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ *be the stationary solution of* (1.18), (1.20) *and* (1.30). *Suppose that the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (1.5) *and* (1.7). *In addition, assume that the initial data* $(\rho_0, j_0) \in H^2(\Omega)$ *satisfies the condition* (1.10) *and the compatibility condition* $\rho_0(0) = \rho_l$, $\rho_0(1) = \rho_r$, $j_{0x}(0) = j_{0x}(1) = 0$. *Then there exists a positive constant* δ_4 *such that if* $\delta + \|(\rho_0 - \tilde{\rho}^0, j_0 - \tilde{j}^0)\|_2 \leqslant \delta_4$, *the initial boundary value problem* (1.4), (1.5), (1.7) *and* (1.29) *has a unique solution* $(\rho^0, j^0, \phi^0)(t, x) \in \mathfrak{X}_2([0, \infty))$. *Moreover, the solution* (ρ^0, j^0, ϕ^0) *verifies the additional regularity* $\phi - \tilde{\phi} \in \mathfrak{X}_2^2([0, \infty))$ *and the decay estimate*

$$\|(\rho^0 - \tilde{\rho}^0, j^0 - \tilde{j}^0)(t)\|_2 + \|(\phi^0 - \tilde{\phi}^0)(t)\|_4 \leqslant C\|(\rho_0 - \tilde{\rho}^0, j_0 - \tilde{j}^0)\|_2 e^{-\alpha_2 t}, \tag{1.32}$$

where C *and* α_2 *are positive constants independent of* t.

In the above lemma, $\mathfrak{X}_2$ and $\mathfrak{X}_2^2$ denote the function spaces defined by

$$\mathfrak{X}_2([0, T]) := \bigcap_{k=0}^{2} C^k([0, T]; H^{2-k}(\Omega)), \qquad \mathfrak{X}_2^2([0, T]) := \bigcap_{k=0}^{2} C^k([0, T]; H^{4-k}(\Omega)),$$

respectively.

It is naturally expected that the solution to (1.1) approaches that to (1.29) as ε tends to zero. To begin with proving this expectation, we consider the convergence of the stationary solutions. Precisely, we show that the stationary solution $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon)$ to the problem (1.17)–(1.20) converges to the stationary solution $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ to the problem (1.18), (1.20) and (1.30) as ε tends to zero. Then, we study the convergence of non-stationary solutions, $(\rho^\varepsilon, j^\varepsilon, \phi^\varepsilon)$ and (ρ^0, j^0, ϕ^0). The former result is summarized in the next lemma.

Lemma 1.6. *Suppose that the same assumptions in Lemmas* 1.1 *and* 1.4 *hold. Let* $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ *be the stationary solution to* (1.18), (1.20) *and* (1.30), *and* $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon)$ *be the stationary solution to* (1.17)–(1.20). *For an arbitrary* ρ_l, *there exists a positive constant* δ_5 *such that if* $\delta + \varepsilon \leqslant \delta_5$, *then the stationary solution* $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon)$ *to* (1.17)–(1.20) *converges to the stationary solution* $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ *to* (1.18), (1.20) *and* (1.30) *as* ε *tends to zero. Precisely,*

$$\|\tilde{\rho}^\varepsilon - \tilde{\rho}^0\|_1 + |\tilde{j}^\varepsilon - \tilde{j}^0| + \|\tilde{\phi}^\varepsilon - \tilde{\phi}^0\|_3 \leqslant C\varepsilon, \tag{1.33}$$

$$\|(\partial_x^2\{\tilde{\rho}^\varepsilon - \tilde{\rho}^0\}, \partial_x^4\{\tilde{\phi}^\varepsilon - \tilde{\phi}^0\}, \varepsilon\partial_x^3\tilde{\rho}^\varepsilon, \varepsilon^2\partial_x^4\tilde{\rho}^\varepsilon)\| \to 0 \quad \text{as } \varepsilon \to 0, \tag{1.34}$$

where the positive constant C *is independent of* ε.

The classical limit of the non-stationary problem is stated in the following theorem.

Theorem 1.7. *Assume that the same conditions in Theorem* 1.2 *and Lemma* 1.5 *hold. Then there exists a positive constant* δ_6 *such that if*

$$\delta + \varepsilon + \left\| \left(\rho_0 - \tilde{\rho}^0, j_0 - \tilde{j}^0 \right) \right\|_2 + \left\| \left(\rho_0 - \tilde{\rho}^\varepsilon, j_0 - \tilde{j}^\varepsilon \right) \right\|_2 \\ + \left\| \left(\varepsilon \partial_x^3 \{ \rho_0 - \tilde{\rho}^\varepsilon \}, \varepsilon \partial_x^3 \{ j_0 - \tilde{j}^\varepsilon \}, \varepsilon^2 \partial_x^4 \{ \rho_0 - \tilde{\rho}^\varepsilon \} \right) \right\| \leqslant \delta_6, \tag{1.35}$$

then the time global solution $(\rho^\varepsilon, j^\varepsilon, \phi^\varepsilon)$ *to* (1.1), (1.4)–(1.7) *approaches the solution* (ρ^0, j^0, ϕ^0) *to* (1.4), (1.5), (1.7) *and* (1.29) *as* ε *tends to zero. Precisely,*

$$\left\| \left(\rho^\varepsilon - \rho^0, j^\varepsilon - j^0 \right)(t) \right\|_1 + \left\| \left(\phi^\varepsilon - \phi^0 \right)(t) \right\|_3 \leqslant \sqrt{\varepsilon} C e^{\beta t} \quad \text{for } t \in [0, \infty), \tag{1.36}$$

$$\sup_{t \in [0,\infty)} \left\{ \left\| \left(\rho^\varepsilon - \rho^0, j^\varepsilon - j^0 \right)(t) \right\|_1 + \left\| \left(\phi^\varepsilon - \phi^0 \right)(t) \right\|_3 \right\} \to 0 \quad \text{as } \varepsilon \to 0, \tag{1.37}$$

where β *and* C *are positive constants independent of* ε *and* t.

Remark 1.8. The convergence of the stationary solution in Lemma 1.6 ensures that we can take the initial data (ρ_0, j_0) verifying the condition (1.35) in Theorem 1.7 if the constant ε is sufficient small.

Related results. Recently a quantum effect, depending on particle resonant tunneling through potential barriers and charge density built-up in quantum wells, becomes important for the analysis of the behavior of electron flow in semiconductor devices as they become truly minute. It also gives a lot of mathematical problems. Here, we state several references on related topics, including the non-quantum model.

The non-quantum hydrodynamic model is introduced by Bløtekjær [2]. Degond and Markowich [3] investigated the stationary solution to the one-dimensional non-quantum model with the Dirichlet boundary condition. They proved the existence of the stationary solution, satisfying the subsonic condition (1.10a). Li, Markowich and Mei [11] studied the asymptotic stability of the stationary solution. Both of papers assume that the doping profile is flat. Matsumura and Murakami [12] proved the asymptotic stability of the stationary solution for the general doping profile. The recent research by Guo and Strauss in [5] solved an important problem, the asymptotic stability of the stationary solution with the Dirichlet boundary condition for the general doping profile. Also, see [13].

More recently, as we have stated, the quantum model draws a lot of attentions of the researchers in semiconductors. The frontier works in mathematics is given by Jüngel and Li [7,8]. The existence of the stationary solution is shown in [7], where they have shown the facts that: *for a given electric current* $\tilde{j}$, *there exists a certain value of the boundary potential* ϕ_r *such that the stationary solution exists.* However, physical experiments are made in order to measure the quantity of the electric current $\tilde{j}$ for the given potential ϕ_r on the boundary. Hence, we reconsider this problem in the second section to cover the important problem in physics and technology. The stability of the stationary solution is proved in [8] under the assumption that the doping profile is flat, that is, $|D(x) - \rho_l| \ll 1$. This assumption is too narrow to solve current problems in technology since typical examples of the doping profile does not satisfy the flatness assumption (see [4]). For instance, the doping profiles of $n^+ - n - n^+$ diodes have two steep slops and seems like a deep valley. However, this flatness assumption makes the problem relatively easy since it makes the stationary solution also flat, i.e. $|(\tilde{\rho}_x, \tilde{\phi}_x)| \ll 1$, and thus the derivation of the a priori estimate easy. Even, without the flatness assumption, we can derive the a priori estimate by using the smallness of the strength of the boundary data δ and the scaled Planck constant ε, which are reasonable from the physical point of view. In this computation, we define and utilize the energy form to handle the large $(\tilde{\rho}_x, \tilde{\phi}_x)$. Moreover, it is also plays an essential role that certain combinations of $\tilde{\rho}_x$ and $\tilde{\phi}_x$ are small.

At last we mention about boundary conditions on the quantum effect. The papers [7,8] adopted the condition that $\rho_x = 0$ on the boundary. On the contrary, we assume the boundary condition (1.6), which implies the quantum (Bohm) potential vanishes on the boundary. Both of boundary conditions for the quantum effect are seriously studied in [4,14]. As far as we know, it is still controversial problem between researches in physics and technology which boundary condition is suitable for the quantum effect.

Outline of the paper. The remaining part of the present paper is organized as follows. In Section 2, we begin detailed discussions with the proof of the existence and the uniqueness of the stationary solution. The existence is proved in Section 2.1 by the Schauder fixed-point theorem. The uniqueness follows from the energy method in Section 2.2. In Section 3.1, we obtain the elliptic estimate and then we establish the unique existence of the time local solution by using an iteration method by solving the non-linear hyperbolic equations. Here we postpone the discussion on the solvability of the linearized hyperbolic problem until Appendix A. Sections 3.2–3.4 are devoted to showing the asymptotic stability of the stationary solution. First, we introduce the energy form to obtain the basic estimate. Next, we derive the system of the equations for the perturbation from the stationary solution. Then an elementary energy method yields the higher order estimates. Therefore, combining the existence of the time local solution and the a priori estimate in the H^2-Sobolev space, we complete the proof of the existence of the time global solution. Finally, by using the uniform estimates previously obtained, we show the exponential convergence of the solution, for the non-stationary problem, to the corresponding stationary solution in Section 3.5. Section 4 is devoted to arguments of the classical limit. We prove that the solution for the quantum model converges to that for the non-quantum model. In this theorem, the convergence of stationary solutions plays important role. Both results are proved by the energy method.

Notation. For a nonnegative integer $l \geqslant 0$, $H^l(\Omega)$ denotes the lth order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. $C^k([0,T]; H^l(\Omega))$ denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\Omega)$. For a nonnegative integer $k \geqslant 0$, $\mathcal{B}^k(\overline{\Omega})$ denotes the space of the functions whose derivatives up to kth order are continuous and bounded over $\overline{\Omega}$, equipped with the norm

$$|f|_k := \sum_{i=0}^{k} \sup_{x \in \overline{\Omega}} \left|\partial_x^i f(x)\right|.$$

Throughout the present paper C and c denote various generic positive constants.

2. Unique existence of stationary solution

2.1. Existence

In this subsection, we discuss the existence and the uniqueness of the stationary solution. Firstly, the existence of the stationary solution is constructed by the Leray–Schauder fixed-point theorem. Secondly, we obtain the estimates of the stationary solution, which is used in showing its uniqueness. Finally, its uniqueness is proved by the energy method.

Apparently, (1.17) is equivalent to (1.17a) and (1.24) if density $\tilde{\omega}$ is positive. Hence once it is shown that the stationary problem (1.22) and (1.24) with the current-voltage relationship (1.25) has a solution $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ satisfying $\tilde{\omega} > 0$, the existence of the solution to the problem (1.17), (1.18)–(1.20) immediately follows. In fact, let $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ be the solution to (1.22) and (1.24) satisfying (1.25). Substituting $x = 0$ and $x = 1$ in (1.24a), we see that $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ verifies the boundary condition (1.19). Equation (1.17b) is obtained by differentiating Eq. (1.24a) and multiplying the resultant equation by $\tilde{\omega}^2$. Moreover, Eq. (1.21b) and the boundary condition (1.20) immediately follow from (1.24b).

The following constants are frequently used in the discussion on the properties of the stationary solution:

$$B_0 := |D|_0 + \phi_r + \sqrt{K} + \frac{K}{2} + K\left|\log\frac{\rho_r}{\rho_l}\right|,$$
$$B_M := \max\{\omega_l,\ \omega_r\}\exp\left(\frac{B_0}{2K}\right), \qquad B_m := \min\{\omega_l,\ \omega_r\}\exp\left(\frac{-1}{2K}\left\{B_M^2 + B_0 + \frac{K}{2}\right\}\right).$$

We also use a function $A(x) := \omega_l(1-x) + \omega_r x$.

Lemma 2.1. *Let the doping profile and the boundary data satisfy conditions* (1.3), (1.5) *and* (1.7). *Moreover, suppose that the following inequalities hold*:

$$B_M^{-4} + 2B_b\left(\rho_r^{-2} - \rho_l^{-2}\right) > 0, \tag{2.1a}$$
$$S\left[B_m^2, \mathcal{J}\left[B_M^2\right]\right] > 0. \tag{2.1b}$$

Then, the stationary problem (1.22) *and* (1.24) *with the current-voltage relationship* (1.25) *has a solution* $(\tilde{\rho}, \tilde{j}, \tilde{\phi}) \in \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ *satisfying the condition* (1.10). *Furthermore, it holds that* $\tilde{j} \lesseqgtr 0$ *if and only if* $B_b \lesseqgtr 0$.

Proof. Note that there exists a positive constant μ such that

$$(B_M + \mu)^{-4} + 2B_b\left(\rho_r^{-2} - \rho_l^{-2}\right) \geqslant 0, \tag{2.2}$$
$$S\left[(B_m - \mu)^2, \mathcal{J}\left[(B_M + \mu)^2\right]\right] > 0 \tag{2.3}$$

owing to the condition (2.1).

Now we define a mapping $T : v \mapsto V$ over H^1 by solving a linear problem

$$\varepsilon^2 V_{xx} = g(v_{\alpha,\beta}),$$

$$g(v_{\alpha,\beta}) := v_{\alpha,\beta}\left(F\left(v_{\alpha,\beta}^2, \mathcal{J}\left[v_{\alpha,\beta}^2\right]\right) - F\left(\rho_l, \mathcal{J}\left[v_{\alpha,\beta}^2\right]\right) - \mathcal{G}\left[v_{\alpha,\beta}^2\right] + \mathcal{J}\left[v_{\alpha,\beta}^2\right]\int_0^x \frac{1}{v_{\alpha,\beta}^2}(y)\,dy\right),$$

$$v_{\alpha,\beta} := \max\{\beta, \min\{\alpha, v\}\}, \qquad \alpha := B_M + \mu, \qquad \beta := B_m - \mu \tag{2.4}$$

with boundary condition (1.22), where F, $\mathcal{J}$ and $\mathcal{G}$ are given in (1.24c), (1.27) and (1.24b), respectively.

Notice that the constant $\mathcal{J}[v_{\alpha,\beta}^2]$ is defined by solving the current-voltage relationship (1.25) with $(v_{\alpha,\beta}^2, \mathcal{J}[v_{\alpha,\beta}^2])$ in place of $(\tilde{\omega}^2, \tilde{j})$ due to (2.2). Apparently, the mapping T is well defined by the standard theory of the elliptic equations. In fact, $g(v_{\alpha,\beta})$ belongs to H^1 owing to $v_{\alpha,\beta} \in H^1$. Thus we have the solution $T(v) = V \in H^3$ to the problem (2.4) and (1.22). In addition, the mapping T is a continuous and compact mapping from H^1 into itself. Next, in order to apply the Leray–Schauder fixed-point theorem, we show that there exists a positive constant M such that $\|u\|_1 \leqslant M$ for an arbitrary $u \in \{f \in H^1;\ f = \lambda T(f) \text{ for } \lambda \in [0,1]\}$. We may assume $\lambda > 0$ as the case $\lambda = 0$ is trivial. Here it is sufficient to show that $\|\tilde{\omega}\|_1 \leqslant M$ for $\tilde{\omega}$ satisfying an equation and a boundary condition,

$$\varepsilon^2 \tilde{\omega}_{xx} = \lambda g(\tilde{\omega}_{\alpha,\beta}), \tag{2.5}$$

$$\tilde{\omega}(0) = \lambda \omega_l, \qquad \tilde{\omega}(1) = \lambda \omega_r. \tag{2.6}$$

Multiplying (2.5) by $(\tilde{\omega} - \lambda A)$, integrating the resultant equality over the domain Ω and using the estimate $|g(v_{\alpha,\beta})| \leqslant C$, where C is a positive constant depending on α, β, ρ_r, ρ_l, ϕ_r and $|D|_0$, we have the desired estimate $\|\tilde{\omega}\|_1 \leqslant M$. Hence, we see that the mapping T has a fixed point $\tilde{\omega} = T(\tilde{\omega}) \in H^3$ by the Leray–Schauder fixed-point theorem (see Theorem 11.3 in [6] for example). Namely,

$$\varepsilon^2 \tilde{\omega}_{xx} = g(\tilde{\omega}_{\alpha,\beta}). \tag{2.7}$$

It suffices to show $\tilde{\omega} = \tilde{\omega}_{\alpha,\beta}$ for the completion of the proof. Owing to (2.3), the pair of $(\tilde{\omega}_{\alpha,\beta}^2, \mathcal{J}[\tilde{\omega}_{\alpha,\beta}^2])$ satisfies the subsonic condition $S[(\tilde{\omega}_{\alpha,\beta}^2, \mathcal{J}[\tilde{\omega}_{\alpha,\beta}^2])] > 0$. For the case $\rho_l \geqslant \rho_r$, add $-K\tilde{\omega}_{\alpha,\beta}\log\rho_l$ to Eq. (2.7), and multiply the resultant equality by $(\log\tilde{\omega}_{\alpha,\beta}^2 - \log\rho_l)_+^n$ for $n = 1, 2, 3, \ldots$. For the case $\rho_l < \rho_r$, add $-K\tilde{\omega}_{\alpha,\beta}\log\rho_r$ to (2.7), and multiply by $(\log\tilde{\omega}_{\alpha,\beta}^2 - \log\rho_r)_+^n$ for $n = 1, 2, 3, \ldots$. Here we have used the notation: $(\,\cdot\,)_+ := \max\{0, \cdot\}$. From now on we treat the case $\rho_l < \rho_r$ only since the other case is more easily handled. The above computations yield

$$-\varepsilon^2 \tilde{\omega}_{xx}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^n + K\tilde{\omega}_{\alpha,\beta}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^{n+1}$$
$$= \left(\phi_r x - \int_0^1 GD\,d\xi - \int_0^x \frac{\mathcal{J}[\tilde{\omega}_{\alpha,\beta}^2]}{\tilde{\omega}_{\alpha,\beta}^2}\,dy + \frac{\mathcal{J}^2[\tilde{\omega}_{\alpha,\beta}^2]}{2\rho_l^2} + K\log\frac{\rho_l}{\rho_r}\right)\tilde{\omega}_{\alpha,\beta}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^n$$

$$-\left(-\int_0^1 G\tilde{\omega}_{\alpha,\beta}^2\,d\xi+\frac{\mathcal{J}^2[\tilde{\omega}_{\alpha,\beta}^2]}{2\tilde{\omega}_{\alpha,\beta}^4}\right)\tilde{\omega}_{\alpha,\beta}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^n\leqslant B_0\tilde{\omega}_{\alpha,\beta}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^n. \tag{2.8}$$

In the derivation of the above inequality, we have used the subsonic condition (1.10a) and $|G|\leqslant 1$ for the first term of the right-hand side of the equality, and the fact that G is non-positive for the second term. (We remark that $K|\log\rho_l-\log\rho_r|$ in the right-hand side of the inequality (2.8) vanishes for the case $\rho_l\geqslant\rho_r$.) The first term of (2.8) is rewritten as

$$(1\text{st term})=2\varepsilon^2 n\frac{(\tilde{\omega}_{\alpha,\beta_x})^2}{\tilde{\omega}_{\alpha,\beta}}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^{n-1}-\left\{\varepsilon^2\tilde{\omega}_x\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^n\right\}_x. \tag{2.9}$$

By the Young inequality, the last term of (2.8) is estimated as

$$(\text{last term})\leqslant\frac{n}{n+1}K\tilde{\omega}_{\alpha,\beta}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^{n+1}+\frac{|K\tilde{\omega}_{\alpha,\beta}|_0}{(n+1)}\left(\frac{B_0}{K}\right)^{n+1}. \tag{2.10}$$

Note that the first term on the right-hand side of (2.9) is nonnegative, and the second term vanishes after the integration due to $(\log\tilde{\omega}_{\alpha,\beta}^2-\log\rho_r)_+(0)=(\log\tilde{\omega}_{\alpha,\beta}^2-\log\rho_r)_+(1)=0$. Substituting (2.9) and (2.10) in (2.8) and integrating the result, we have

$$\int_0^1 K\sqrt{\rho_r}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^{n+1}dx\leqslant|K\tilde{\omega}_{\alpha,\beta}|_0\left(\frac{B_0}{K}\right)^{n+1}, \tag{2.11}$$

where we have also used $\tilde{\omega}_{\alpha,\beta}\geqslant\sqrt{\rho_r}$. Taking the $(n+1)$th root of (2.11) yields

$$\left(\int_0^1\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_+^{n+1}dx\right)^{1/(n+1)}\leqslant\left(\frac{|\tilde{\omega}_{\alpha,\beta}|_0}{\sqrt{\rho_r}}\right)^{1/(n+1)}\frac{B_0}{K}. \tag{2.12}$$

Letting $n\to\infty$ in (2.12), we have $\tilde{\omega}_{\alpha,\beta}^2\leqslant B_M^2$.

We show the lower bound of $\tilde{\omega}_{\alpha,\beta}^2$. Add $-K\tilde{\omega}_{\alpha,\beta}\log\rho_r$ to Eq. (2.7), and multiply $(\log\tilde{\omega}_{\alpha,\beta}^2-\log\rho_r)_-^{2n-1}/\tilde{\omega}_{\alpha,\beta}$ for $n=1,2,3,\ldots$, by the resultant equality for the case $\rho_l\geqslant\rho_r$. For the case $\rho_l<\rho_r$, add $-K\tilde{\omega}_{\alpha,\beta}\log\rho_l$ to Eq. (2.7), and multiply $(\log\tilde{\omega}_{\alpha,\beta}^2-\log\rho_l)_-^{2n-1}/\tilde{\omega}_{\alpha,\beta}$ for $n=1,2,3,\ldots$, by the resultant equality. Here we have used the notation: $(\,\cdot\,)_-:=\min\{0,\cdot\}$. We treat the former case only since the latter case is easier. The above computations yield that

$$\begin{aligned}
&-\varepsilon^2\frac{\tilde{\omega}_{xx}}{\tilde{\omega}_{\alpha,\beta}}\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_-^{2n-1}+K\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_-^{2n}\\
&\quad=\left(\mathcal{G}[\tilde{\omega}_{\alpha,\beta}^2]-\int_0^x\frac{\mathcal{J}[\tilde{\omega}_{\alpha,\beta}^2]}{2\tilde{\omega}_{\alpha,\beta}^2}dy+\frac{\mathcal{J}^2[\tilde{\omega}_{\alpha,\beta}^2]}{2\rho_l^2}-\frac{\mathcal{J}^2[\tilde{\omega}_{\alpha,\beta}^2]}{2\tilde{\omega}_{\alpha,\beta}^4}+K\log\frac{\rho_l}{\rho_r}\right)\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_-^{2n-1}\\
&\quad\leqslant\frac{2n-1}{2n}K\left(\log\frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r}\right)_-^{2n}+\frac{K}{2n}\left(\frac{B_M^2}{K}+\frac{B_0}{K}+\frac{1}{2}\right)^{2n}.
\end{aligned}\tag{2.13}$$

In deriving the inequality in (2.13), we have used subsonic condition (1.10a), the inequality $|G| \leqslant 1$ and the Young inequality. Rewrite the first term in (2.13) as

$$(\text{1st term}) = \varepsilon^2 \frac{(\tilde{\omega}_{\alpha,\beta_x})^2}{\tilde{\omega}_{\alpha,\beta}^2} \left\{ 2(2n-1) \left(\log \frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r} \right)_-^{2n-2} - \left(\log \frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r} \right)_-^{2n-1} \right\}$$
$$- \left\{ \frac{\varepsilon^2 \tilde{\omega}_x}{\tilde{\omega}_{\alpha,\beta}} \left(\log \frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r} \right)_-^{2n-1} \right\}_x. \tag{2.14}$$

The first term in (2.14) is nonnegative. The last term in (2.14) disappears after the integration since $(\log \tilde{\omega}_{\alpha,\beta}^2 - \log \rho_r)_-(0) = (\log \tilde{\omega}_{\alpha,\beta}^2 - \log \rho_r)_-(1) = 0$. Substituting (2.14) in (2.13), integrating the result over Ω and then taking $2n$th root yield

$$\left(\int_0^1 \left(\log \frac{\tilde{\omega}_{\alpha,\beta}^2}{\rho_r} \right)_-^{2n} dx \right)^{1/2n} \leqslant \frac{1}{K} \left(B_M^2 + B_0 + \frac{K}{2} \right). \tag{2.15}$$

We have $B_m^2 \leqslant \tilde{\omega}_{\alpha,\beta}^2$ by letting $n \to \infty$ in (2.15).

Consequently, we have shown $B_m \leqslant \tilde{\omega}_{\alpha,\beta} \leqslant B_M$, which means $\tilde{\omega} = \tilde{\omega}_{\alpha,\beta}$. Hence $(\tilde{\omega}, \mathcal{J}[\tilde{\omega}^2], \mathcal{G}[\tilde{\omega}^2])$ is a solution to the problem (1.22) and (1.24) with (1.25). Differentiate Eq. (1.24a) and use the regularity $\tilde{\omega} \in H^3$ to obtain the desired regularity of the stationary solution. Furthermore, we see from (1.27) that $\mathcal{J}[\tilde{\omega}^2] \lesseqqgtr 0$ holds if and only if $B_b \lesseqqgtr 0$. □

2.2. Uniqueness

Lemma 2.1 ensures the existence of the stationary solution. In order to show its uniqueness, we need an additional assumption (see Lemma 2.4). We prove several estimates for the stationary solution in Lemma 2.2 before discussing its uniqueness. The following inequalities are frequently used in the proof of Lemma 2.2:

$$|f|_0^2 \leqslant \|f\|^2 + 2\|f\|\|f_x\| \quad \text{for } f \in H^1(\Omega), \tag{2.16}$$

$$\|f\|^2 \leqslant \frac{1}{4}\|f_x\|^2 \quad \text{for } f \in H_0^1(\Omega), \tag{2.17}$$

$$\|f_x\|^2 \leqslant \frac{1}{2}\|f_{xx}\|^2 \quad \text{for } f \in \left\{ f \in H^2(\Omega);\ f(0) = f(1) \right\}. \tag{2.18}$$

Lemma 2.2. *Let $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ be a stationary solution in $\mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ to the problem* (1.17a)*,* (1.20) *and* (1.21)–(1.23) *satisfying the condition* (1.10)*. Assume the conditions in* (2.1) *and the inequality*

$$\sqrt{K} < \left| 2B_b \mathcal{J}\left[B_M^2\right]^{-1} \left(\rho_r^{-2} - \rho_l^{-2}\right)^{-1} B_M^{-2} \right| \tag{2.19}$$

hold. Then the solution $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$ verifies (1.27) *and the followings*:

$$B_m \leqslant \tilde{\omega} \leqslant B_M, \tag{2.20}$$

$$|\tilde{\phi}|_2 \leqslant C, \tag{2.21}$$

$$\|\tilde{\omega}\|_2 \leqslant C, \qquad \left\|\partial_x^3 \tilde{\omega}\right\| \leqslant C\varepsilon^{-1} + C, \qquad \left\|\partial_x^4 \tilde{\omega}\right\| \leqslant C\varepsilon^{-2} + C, \tag{2.22}$$

where C is a positive constant depending only on ρ_l, ρ_r, ϕ_r and $|D|_0$ but independent of ε.

Proof. The estimate (2.20) is proved similarly as the derivation of $\tilde{\omega}_{\alpha,\beta} = \tilde{\omega}$ in Lemma 2.1. The estimate (2.21) follows from the formula (1.24b) with the aid of the estimate (2.20). By solving the current-voltage relationship (1.25) with respect to $\tilde{j}$, we see the solution $\tilde{j}$ is given by (1.27). The other solution violates the subsonic condition (1.10a) thanks to (2.19), although (1.25) is the quadratic equation.

It suffices to show (2.22) for the completion of the proof. Multiply (1.21a) by $\tilde{\omega}_x/\tilde{\omega}^2$ and integrate the resultant equality over Ω by parts with using the boundary conditions (1.22) and (1.23) as well as the equality (1.21b) to obtain that

$$\begin{aligned}
&\int_0^1 2S\big[\tilde{\omega}^2, \tilde{j}\big]\frac{\tilde{\omega}_x^2}{\tilde{\omega}} + \varepsilon^2 \frac{\tilde{\omega}_{xx}^2}{\tilde{\omega}}\, dx \\
&\quad = \int_0^1 -\big(\tilde{\omega}^2 - D\big)(\tilde{\omega} - A) + \tilde{\phi}_x A_x\, dx + \tilde{j}\left(\frac{1}{\tilde{\omega}_r} - \frac{1}{\tilde{\omega}_l}\right) \leqslant C,
\end{aligned} \tag{2.23}$$

where C is a positive constant depending only on ρ_l, ρ_r, ϕ_r and $|D|_0$ but independent of ε. In deriving the above inequality, we have also used the estimates (2.20) and (2.21). Since the left-hand side of (2.23) is estimated by $2S[B_m^2, \mathcal{J}[B_M^2]]\|\tilde{\omega}\|^2/B_M$ below, we get $\|\tilde{\omega}_x\| \leqslant C$.

Multiply (1.21a) by $(\tilde{\omega}_{xx}/\tilde{\omega})_x/\tilde{\omega}^2$, integrate the resultant equality over Ω, apply the integration by parts and then use (1.23) and (1.21b). The result is

$$\begin{aligned}
&\int_0^1 \varepsilon^2\left(\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)_x^2 + 2S\big[\tilde{\omega}^2, \tilde{j}\big]\left(\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)^2 dx \\
&\quad = \int_0^1 -2\left(S\big[\tilde{\omega}^2, \tilde{j}\big]\frac{1}{\tilde{\omega}}\right)_x \frac{\tilde{\omega}_x \tilde{\omega}_{xx}}{\tilde{\omega}}\, dx + \int_0^1 \left\{\big(\tilde{\omega}^2 - D\big) - \left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)_x\right\}\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\, dx.
\end{aligned} \tag{2.24}$$

The first term of the right-hand side of (2.24) is estimated by the Hölder and the Schwarz inequalities, and (2.16) as

$$\begin{aligned}
\big|(\text{1st term})\big| &\leqslant C|\tilde{\omega}_x|_0\|\tilde{\omega}_x\|\|\tilde{\omega}_{xx}\| \\
&\leqslant C\sqrt{\|\tilde{\omega}_x\|^2 + 2\|\tilde{\omega}_x\|\|\tilde{\omega}_{xx}\|}\,\|\tilde{\omega}_x\|\|\tilde{\omega}_{xx}\| \\
&\leqslant C\big(1 + \|\tilde{\omega}_{xx}\|\big) + \frac{S[B_m^2, \mathcal{J}[B_M^2]]}{B_M^2}\|\tilde{\omega}_{xx}\|^2.
\end{aligned}$$

The inequalities (2.17) and (2.20) together with the Hölder inequality give $|(\text{2nd term})| \leqslant C(1 + \|\tilde{\omega}_{xx}\|)$. Note that the left-hand side of (2.24) is estimated by $2S[B_m^2, \mathcal{J}[B_M^2]]\|\tilde{\omega}_{xx}\|^2/B_M^2$ below. Substitute these estimates in (2.24) and solve the resultant inequality with respect to $\|\tilde{\omega}_{xx}\|$ to obtain $\|\tilde{\omega}_{xx}\| \leqslant C$. Hence the first inequality in (2.22) is proved.

Then we show the second inequality in (2.22). Substituting these three inequalities in (2.24) gives the estimate $\varepsilon^2\|(\tilde{\omega}_{xx}/\tilde{\omega})_x\|^2 \leqslant C$. Note that the following estimate is obtained from the estimates (2.16) and (2.20): $\|(\tilde{\omega}_{xx}/\tilde{\omega})_x\| \geqslant \|\tilde{\omega}_{xxx}\|/B_M - C$. Owing to these two inequalities, we have $\|\tilde{\omega}_{xxx}\| \leqslant C + C/\varepsilon$, which is the second estimate in (2.22). Furthermore, differentiate Eq. (1.21a) and multiply the resultant equality by $1/\tilde{\omega}$. The result is

$$\varepsilon^2\tilde{\omega}_{xxxx} = \varepsilon^2\frac{\tilde{\omega}_{xx}^2}{\tilde{\omega}} + \frac{2}{\tilde{\omega}}\big(S\big[\tilde{\omega}^2, \tilde{j}\big]\tilde{\omega}\tilde{\omega}_x\big)_x - 2\tilde{\omega}_x\tilde{\phi}_x - \tilde{\omega}\tilde{\phi}_{xx}. \tag{2.25}$$

Estimating the right-hand side of (2.25) with using (2.16) and (2.20), we have $\|\tilde{\omega}_{xxxx}\| \leqslant C + C/\varepsilon^2$, which is the third estimate in (2.22). □

Apparently, the next corollary follows from the proof of Lemma 2.2.

Corollary 2.3. *Assume the same conditions as in Lemma* 2.2*. For an arbitrary ρ_l, there exists a positive constant δ_1 such that if $\delta + \varepsilon \leqslant \delta_1$, then the stationary solution satisfies the estimates* (2.21) *and* (2.22) *with a positive constant C depending only on ρ_l and $|D|_0$ but independent of δ and ε.*

We are at the position to consider the uniqueness of the stationary solution. Let $\tilde{w} := \log\tilde{\rho} = \log\tilde{\omega}^2$ and rewrite (1.17)–(1.20) for $(\tilde{w}, \tilde{j}, \tilde{\phi})$ as

$$\tilde{j}_x = 0, \tag{2.26a}$$

$$S\big[e^{\tilde{w}}, \tilde{j}\big]\tilde{w}_x - \tilde{\phi}_x - \frac{\varepsilon^2}{2}\left(\tilde{w}_{xx} + \frac{\tilde{w}_x^2}{2}\right)_x = -\frac{\tilde{j}}{e^{\tilde{w}}}, \tag{2.26b}$$

$$\tilde{\phi}_{xx} = e^{\tilde{w}} - D, \tag{2.26c}$$

$$\tilde{w}(0) = \log\rho_l, \qquad \tilde{w}(1) = \log\rho_r, \tag{2.27}$$

$$\left(\tilde{w}_{xx} + \frac{\tilde{w}_x^2}{2}\right)(0) = \left(\tilde{w}_{xx} + \frac{\tilde{w}_x^2}{2}\right)(1) = 0, \tag{2.28}$$

$$\tilde{\phi}(0) = 0, \qquad \tilde{\phi}(1) = \phi_r > 0. \tag{2.29}$$

Note that if the uniqueness of the stationary solution to (2.26)–(2.29) with $\tilde{\rho} > 0$ is proved, the uniqueness of that to (1.17)–(1.20) immediately follows.

Lemma 2.4. *Assume the same conditions in Lemma* 2.2*. For an arbitrary ρ_l, there exists a positive constant δ_1 such that if $\delta + \varepsilon \leqslant \delta_1$, then the solution $(\tilde{w}, \tilde{j}, \tilde{\phi})$ in $\mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^4(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$, satisfying* (1.10)*, is unique.*

Proof. Owing to Lemma 2.2, $\tilde{j}$ is written by the explicit formula (1.27), i.e., $\tilde{j} = \mathcal{J}[e^{\tilde{w}}]$. Let $(\tilde{w}_1, \tilde{j}_1, \tilde{\phi}_1)$ and $(\tilde{w}_2, \tilde{j}_2, \tilde{\phi}_2)$ be solutions to the stationary problem (2.26)–(2.29). Taking the difference $\tilde{j}_1 = \mathcal{J}[e^{\tilde{w}_1}]$ and $\tilde{j}_2 = \mathcal{J}[e^{\tilde{w}_2}]$, and then using the mean value theorem and (2.20), we have

$$|\tilde{j}_1 - \tilde{j}_2| \leqslant C\delta \|\tilde{w}_1 - \tilde{w}_2\|, \tag{2.30}$$

where C is a positive constant independent of δ and ε. Due to (2.26b), the difference $\bar{w} := \tilde{w}_1 - \tilde{w}_2$ satisfies

$$\begin{aligned} &-\frac{\varepsilon^2}{2}\left(\bar{w}_{xx} + \frac{\tilde{w}_{1x}^2}{2} - \frac{\tilde{w}_{2x}^2}{2}\right)_x + S\bigl[e^{\tilde{w}_1}, \tilde{j}_1\bigr]\bar{w}_x - (\phi_1 - \phi_2)_x \\ &\quad = \left(\frac{\tilde{j}_1^2}{e^{2\tilde{w}_1}} - \frac{\tilde{j}_2^2}{e^{2\tilde{w}_2}}\right)\tilde{w}_{2x} - \left(\frac{\tilde{j}_1}{e^{\tilde{w}_1}} - \frac{\tilde{j}_2}{e^{\tilde{w}_2}}\right). \end{aligned} \tag{2.31}$$

Multiply Eq. (2.31) by $\bar{w}_x$, integrate the resultant equality and use the boundary conditions (2.28) and (2.29) as well as Eq. (2.26c) to obtain that

$$\begin{aligned} &\int_0^1 \frac{\varepsilon^2}{2}\bar{w}_{xx}^2 + S\bigl[e^{\tilde{w}_1}, \tilde{j}_1\bigr]\bar{w}_x^2 + \bigl(e^{\tilde{w}_1} - e^{\tilde{w}_2}\bigr)\bar{w}\,dx \\ &\quad = \int_0^1 \left\{\left(\frac{\tilde{j}_1}{e^{\tilde{w}_1}} + \frac{\tilde{j}_2}{e^{\tilde{w}_2}}\right)\tilde{w}_{2x} - 1\right\}\left(\frac{\tilde{j}_1}{e^{\tilde{w}_1}} - \frac{\tilde{j}_2}{e^{\tilde{w}_2}}\right)\bar{w}_x\,dx \\ &\qquad - \int_0^1 \frac{\varepsilon^2}{4}(\tilde{w}_1 + \tilde{w}_2)_x \bar{w}_x \bar{w}_{xx}\,dx. \end{aligned} \tag{2.32}$$

We handle the first term of the right-hand side of (2.32) by using the estimates (2.16), (2.17), (2.20) and (2.30) as

$$\bigl|(\text{1st term})\bigr| \leqslant C\bigl(\bigl\|\tilde{j}_1\bigl(e^{-\tilde{w}_1} - e^{-\tilde{w}_2}\bigr)\bigr\| + \bigl\|e^{-\tilde{w}_2}(\tilde{j}_1 - \tilde{j}_2)\bigr\|\bigr)\|\bar{w}_x\| \leqslant C\delta\|\bar{w}_x\|^2. \tag{2.33}$$

The second term of the right-hand side of (2.32) is estimated by the Schwarz and the Hölder inequalities as

$$\bigl|(\text{2nd term})\bigr| \leqslant \varepsilon^2 C\|\bar{w}_x\|\|\bar{w}_{xx}\| \leqslant \frac{\varepsilon^2}{2}\|\bar{w}_{xx}\|^2 + C\varepsilon^2\|\bar{w}_x\|^2, \tag{2.34}$$

where we have used (2.16) and Corollary 2.3. Substituting the equalities (2.33) and (2.34) in (2.32), we see from letting δ and ε small enough that $\|\bar{w}\|^2 \leqslant 0$ thanks to the estimates $(e^{\tilde{w}_1} - e^{\tilde{w}_2})\bar{w} \geqslant 0$ and $S[e^{\tilde{w}_1}, \tilde{j}_1] \geqslant S[B_m^2, \mathcal{J}[B_M^2]] > 0$. Thus we have shown $\tilde{w}_1 \equiv \tilde{w}_2$. The equalities $\tilde{j}_1 \equiv \tilde{j}_2$ and $\tilde{\phi}_1 \equiv \tilde{\phi}_2$ immediately follow from (2.26c), (2.29) and (2.30). The proof is completed. □

Consequently, Lemma 1.1 holds apparently from Lemmas 2.1 and 2.4 since the smallness of $\delta + \varepsilon$ implies that all the assumptions in Lemmas 2.1 and 2.4 hold.

3. Asymptotic stability of the stationary solution

3.1. An existence of a solution locally in time

In this subsection, we show the unique existence of the solution locally in time to the initial boundary value problem (1.12)–(1.16) since the problem (1.12)–(1.16) is equivalent to (1.1) and (1.4)–(1.7) with $\rho > 0$. In the following discussion, we follow the ideas in the paper [9,10] which have shown the existence of a time-local solution for the hyperbolic–elliptic coupled systems. Also, see [13].

Lemma 3.1. *Suppose the initial data* $(\omega_0, j_0) \in H^4(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (1.5), (1.7) *and* $\omega_0 > 0$. *Then there exists a constant* $T_1 > 0$ *such that the initial boundary value problem* (1.12)–(1.16) *has a unique solution* $(\omega, j, \phi) \in \overline{\mathfrak{X}}_4([0, T_1]) \times \overline{\mathfrak{X}}_3([0, T_1]) \times \mathfrak{Y}([0, T_1])$ *satisfying* $\omega > 0$.

The next corollary immediately follows from Lemma 3.1. Namely, non-stationary problem has a unique solution satisfying the condition (1.10).

Corollary 3.2. *Suppose the initial data* $(\rho_0, j_0) \in H^4(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (1.5), (1.7), (1.8) *and* (1.11). *Then there exists a constant* $T_2 > 0$ *such that the initial boundary value problem* (1.1) *and* (1.4)–(1.7) *has a unique solution* $(\rho, j, \phi) \in \overline{\mathfrak{X}}_4([0, T_2]) \times \overline{\mathfrak{X}}_3([0, T_2]) \times \mathfrak{Y}([0, T_2])$ *satisfying the condition* (1.10).

We define the successive approximation sequence for solving the problem (1.12)–(1.16). For this purpose, we consider the linearized system for the unknown $(\hat{\omega}, \hat{j})$:

$$2\omega\hat{\omega}_t + \hat{j}_x = 0, \tag{3.1a}$$

$$\hat{j}_t + 2S\big[\omega^2, j\big]\omega\hat{\omega}_x + 2\frac{j}{\omega^2}\hat{j}_x - \varepsilon^2\omega^2\left(\frac{\hat{\omega}_{xx}}{\omega}\right)_x = \omega^2\phi_x - j \tag{3.1b}$$

with the initial data (1.13) and the boundary data (1.14) and (1.15), where the function ϕ is defined by (1.9), i.e., $\phi = \Phi[\omega^2]$. Let the functions (ω, j) in the coefficients in (3.1) satisfy

$$(\omega, j) \in \overline{\mathfrak{X}}_4\big([0, T]\big) \times \overline{\mathfrak{X}}_3\big([0, T]\big), \qquad (\omega, j)(0, x) = (\omega_0, j_0), \tag{3.2}$$

$$\omega(t, x) \geqslant m \quad \text{for } (t, x) \in [0, T] \times \Omega, \tag{3.3}$$

$$\big\|\omega(t)\big\|_4 + \big\|\omega_t(t)\big\|_2 + \big\|\omega_{tt}(t)\big\| + \big\|j(t)\big\|_3 + \big\|j_t(t)\big\|_1 \leqslant M \quad \text{for } t \in [0, T], \tag{3.4}$$

where T, m and M are positive numbers. We denote by $X(T; m, M)$ the set of functions (ω, j) satisfying (3.2)–(3.4), and we abbreviate $X(T; m, M)$ by $X(\cdot)$ without confusion. The property of ϕ is that

$$\phi \in \mathfrak{Y}\big([0, T]\big), \qquad \big\|\partial_t^i\phi(t)\big\|_2 \leqslant M \quad \text{for } i = 0, 1, 2,\ t \in [0, T].$$

Then the next lemma means that for suitably chosen constants T, m and M, the set $X(\cdot)$ is invariant under the mapping $(\omega, j) \to (\hat{\omega}, \hat{j})$ defined by solving the problem (3.1) and (1.13)–(1.15). We discuss the solvability of this linear problem in Appendix A.

Lemma 3.3. *Suppose that the initial data* $(\omega_0, j_0) \in H^4(\Omega) \times H^3(\Omega)$ *and the boundary data* ρ_l *and* ρ_r *satisfy* (1.5) *and* $\omega_0 > 0$ *. In addition, assume the compatibility conditions* (1.8) *hold. Then there exist positive constants* T, m *and* M *satisfying the following property: If* $(\omega, j) \in X(\cdot)$, *then the problem* (1.9), (1.13)–(1.15) *and* (3.1) *admits a unique solution* $(\hat{\omega}, \hat{j})$ *in the same set* $X(\cdot)$.

As this lemma follows from the standard energy method, we omit the details. We can show Lemma 3.1 by using this lemma.

Proof of Lemma 3.1. Define the successive approximation sequence $\{(\omega^n, j^n)\}_{n=0}^{\infty}$ by solving $(\omega^0, j^0) = (\omega_0, j_0)$ and

$$2\omega^n \omega_t^{n+1} + j_x^{n+1} = 0, \tag{3.5a}$$

$$j_t^{n+1} + 2S\big[(\omega^n)^2, j^n\big]\omega^n \omega_x^{n+1} + 2\frac{j^n}{(\omega^n)^2} j_x^{n+1} - \varepsilon^2 (\omega^n)^2 \left(\frac{\omega_{xx}^{n+1}}{\omega^n}\right)_x = (\omega^n)^2 \phi_x^n - j^n, \tag{3.5b}$$

$$\phi^n = \Phi\big[(\omega^n)^2\big] \tag{3.5c}$$

with the initial and the boundary conditions

$$\big(\omega^{n+1}, j^{n+1}\big)(0, x) = (\omega_0, j_0)(x), \tag{3.6}$$

$$\omega^{n+1}(t, 0) = \omega_l, \qquad \omega^{n+1}(t, 1) = \omega_r, \tag{3.7}$$

$$\omega_{xx}^{n+1}(t, 0) = \omega_{xx}^{n+1}(t, 1) = 0 \tag{3.8}$$

for $n = 0, 1, \ldots,$ where Φ is defined in (1.9). Lemma 3.3 implies the sequence $\{(\omega^n, j^n)\}$ is well defined and satisfies $(\omega^n, j^n) \in X(\cdot)$. Moreover, the estimate

$$\big\|\omega^n(t)\big\|_4 + \big\|\omega_t^n(t)\big\|_2 + \big\|\omega_{tt}^n(t)\big\| + \big\|j^n(t)\big\|_3 + \big\|j_t^n(t)\big\|_1 \leqslant M$$

holds for $t \in [0, T]$. Therefore, applying the standard energy method to the linear system of the equations for the difference $(\omega^{n+1} - \omega^n, j^{n+1} - j^n)$, we see that $\{(\omega^n, j^n)\}$ is the Cauchy sequence in $\overline{\mathfrak{X}}_2([0, T]) \times \overline{\mathfrak{X}}_1([0, T])$. In these computations, to obtain the estimates for the higher order derivatives, we estimate the derivatives in time variable t and then rewrite them into those in spatial variable x by using the linear equations. Consequently, there exists a function $(\omega, j) \in \overline{\mathfrak{X}}_2([0, T]) \times \overline{\mathfrak{X}}_1([0, T])$ such that $(\omega^n, j^n) \to (\omega, j)$ strongly in $\overline{\mathfrak{X}}_2([0, T]) \times \overline{\mathfrak{X}}_1([0, T])$ as n tends to infinity. Moreover, it holds $(\omega, j) \in \overline{\mathfrak{X}}_4([0, T]) \times \overline{\mathfrak{X}}_3([0, T])$ by the standard argument (see [15] for example). For the function ω thus obtained, define $\phi := \Phi[\omega^2]$ as (1.9). It is easy to see that (ω, j, ϕ) is the desired solution to the problem (1.12)–(1.16) with $\omega > 0$. Thus the proof of Lemma 3.1 is completed. □

3.2. A priori estimate

To show the asymptotic stability of the stationary solution $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$, we introduce a perturbation from the stationary solution $(\tilde{\omega}, \tilde{j}, \tilde{\phi})$:

$$\psi(t, x) := \omega(t, x) - \tilde{\omega}(x), \qquad \eta(t, x) := j(t, x) - \tilde{j}(x), \qquad \sigma(t, x) := \phi(t, x) - \tilde{\phi}(x).$$

Dividing (1.1b) by ω^2, we have

$$\left(\frac{j}{\omega^2}\right)_t + \frac{j}{\omega^2}\left(\frac{j}{\omega^2}\right)_x + K\left(\log \omega^2\right)_x - \varepsilon^2\left(\frac{\omega_{xx}}{\omega}\right)_x = \phi_x - \frac{j}{\omega^2}. \tag{3.9}$$

Similarly, we obtain from (1.17b) that

$$\frac{\tilde{j}}{\tilde{\omega}^2}\left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)_x + K\left(\log \tilde{\omega}^2\right)_x - \varepsilon^2\left(\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)_x = \tilde{\phi}_x - \frac{\tilde{j}}{\tilde{\omega}^2}. \tag{3.10}$$

Subtracting (1.17a) from (1.12a), (3.10) from (3.9), and (1.17c) from (1.12c), respectively, we derive the equations for the perturbation (ψ, η, σ) as

$$2(\psi + \tilde{\omega})\psi_t + \eta_x = 0, \tag{3.11a}$$

$$\begin{aligned} &\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)_t + \frac{1}{2}\left\{\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)^2 - \left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)^2\right\}_x + K\left(\log(\psi+\tilde{\omega})^2 - \log\tilde{\omega}^2\right)_x \\ &\quad - \varepsilon^2\left(\frac{(\psi+\tilde{\omega})_{xx}}{\psi+\tilde{\omega}} - \frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)_x - \sigma_x + \frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2} - \frac{\tilde{j}}{\tilde{\omega}^2} = 0, \end{aligned} \tag{3.11b}$$

$$\sigma_{xx} = (\psi + 2\tilde{\omega})\psi. \tag{3.11c}$$

The initial and the boundary conditions to the system (3.11) are derived from (1.13)–(1.16) and (1.18)–(1.20) as

$$\psi(x,0) = \psi_0(x) := \omega_0(x) - \tilde{\omega}(x), \qquad \eta(x,0) = \eta_0(x) := j_0(x) - \tilde{j}(x), \tag{3.12}$$

$$\psi(t,0) = \psi(t,1) = 0, \tag{3.13}$$

$$\psi_{xx}(t,0) = \psi_{xx}(t,1) = 0, \tag{3.14}$$

$$\sigma(t,0) = \sigma(t,1) = 0. \tag{3.15}$$

Since $(\tilde{\omega}, \tilde{j}, \tilde{\phi}) \in \overline{\mathfrak{X}}_4([0,T]) \times \overline{\mathfrak{X}}_3([0,T]) \times \mathfrak{Y}([0,T])$ and σ satisfies (3.11c), the local existence of the solution (ψ, η, σ) to the initial boundary value problem (3.11)–(3.15) follows from Lemma 1.1 and Corollary 3.3.

Corollary 3.4. *Suppose that the initial data (ψ_0, η_0) belongs to $H^4(\Omega) \times H^3(\Omega)$ and $((\tilde{\omega} + \psi_0)^2, \tilde{j} + \eta_0)$ satisfy the condition* (1.10). *Then there exists a constant $T_3 > 0$, such that the initial boundary value problem* (3.11)–(3.15) *has a unique local solution $(\psi, \eta, \sigma) \in \overline{\mathfrak{X}}_4([0, T_3]) \times \overline{\mathfrak{X}}_3([0, T_3]) \times \overline{\mathfrak{X}}_4^2([0, T_3])$ with the property that $((\tilde{\omega} + \psi)^2, \tilde{j} + \eta)$ satisfies* (1.10).

Owing to Corollary 3.4, it is sufficient to obtain an a priori estimate (3.16) in order to prove the existence of the solution globally in time. For this purpose, it is convenient to use notations

$$N_\varepsilon(t) := \sup_{0\leqslant\tau\leqslant t} n_\varepsilon(\tau), \qquad n_\varepsilon^2(\tau) := \|(\psi,\eta)(\tau)\|_2^2 + \|(\varepsilon\partial_x^3\psi, \varepsilon\partial_x^3\eta, \varepsilon^2\partial_x^4\psi)(\tau)\|^2,$$

$$M^2(t) := \int_0^t \|(\psi,\eta)(\tau)\|_1^2 + \|\sigma_x(\tau)\|^2\, d\tau.$$

Proposition 3.5. *Let* $(\psi,\eta,\sigma)(t,x) \in \overline{\mathfrak{X}}_4([0,T]) \times \overline{\mathfrak{X}}_3([0,T]) \times \overline{\mathfrak{X}}_4^2([0,T])$ *be a solution to* (3.11)–(3.15). *Then there exists a positive constant* δ_0 *such that if* $N_\varepsilon(T) + \delta + \varepsilon \leqslant \delta_0$, *then the following estimate holds for* $t \in [0,T]$,

$$n_\varepsilon^2(t) + \|\sigma(t)\|_4^2 + \int_0^t n_\varepsilon^2(\tau) + \|\sigma(\tau)\|_4^2\, d\tau \leqslant C n_\varepsilon^2(0), \tag{3.16}$$

where C *is a positive constant independent of* T *and* ε.

3.3. Basic estimate

In order to show the basic estimate, we employ an energy form $\mathcal{E}$ as

$$\mathcal{E} := \frac{1}{2\omega^2}(j-\tilde{j})^2 + \Psi(\omega^2,\tilde{\omega}^2) + \frac{1}{2}\{(\phi-\tilde{\phi})_x\}^2 + \varepsilon^2(\omega-\tilde{\omega})_x^2,$$

$$\Psi(\omega^2,\tilde{\omega}^2) := K\int_{\tilde{\omega}^2}^{\omega^2} \log\xi - \log\tilde{\omega}^2\, d\xi. \tag{3.17}$$

Notice that $\mathcal{E}$ is equivalent to $|(\psi,\eta,\sigma_x,\varepsilon\psi_x)|^2$ if $|(\psi,\eta,\omega_x,\varepsilon\psi_x)| < c$. Namely, there exist positive constants c_1 and C_1 such that

$$c_1|(\psi,\eta,\sigma,\varepsilon\psi_x)|^2 \leqslant \mathcal{E} \leqslant C_1|(\psi,\eta,\sigma,\varepsilon\psi_x)|^2 \tag{3.18}$$

if $|(\psi,\eta,\sigma,\varepsilon\psi_x)| \leqslant c$. Multiply Eq. (3.11b) by η, and apply the integration by parts to obtain the equation of the energy form $\mathcal{E}$:

$$\mathcal{E}_t + \frac{1}{\tilde{\omega}^2}\eta^2 = R_{1x} + R_2,$$

$$R_1 := \sigma\sigma_{xt} + \sigma\eta - K(\log\omega^2 - \log\tilde{\omega}^2)\eta + \varepsilon^2\left(\frac{\omega_{xx}}{\omega} - \frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)\eta + \varepsilon^2\psi_x\psi_t,$$

$$R_2 := \left(\frac{\eta}{2\omega^4} - \frac{j}{\omega^4}\right)\eta\eta_x - \frac{1}{2}\left\{\left(\frac{j}{\omega^2}\right)^2 - \left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)^2\right\}_x\eta + \frac{j(\omega+\tilde{\omega})}{\omega^2\tilde{\omega}^2}\psi\eta + \frac{\varepsilon^2\tilde{\omega}_{xx}}{\tilde{\omega}\omega}\psi\eta_x, \tag{3.19}$$

where we have also used (3.11a) and (3.11c). By applying the inequality (2.16) on R_2 with (1.27), (2.20) and Corollary 2.3, we obtain the estimate

$$|R_2| \leqslant C(N_\varepsilon(T) + \delta + \varepsilon^{3/2})|(\psi,\eta,\psi_x,\eta_x,\sigma_x)|^2. \tag{3.20}$$

Since the next lemma is proved similarly as Lemma 3.3 in [13], we omit the proof.

Lemma 3.6. *Suppose the same assumptions as in Proposition* 3.5 *hold. Then the following estimates hold for* $t \in [0, T]$:

$$\left\|\partial_t^i \sigma(t)\right\|_2^2 \leqslant C\left\|\partial_t^i \psi(t)\right\|^2 \quad \text{for } i = 0, 1, 2, \tag{3.21}$$

$$\left\|\sigma_{xt}(t)\right\|^2 \leqslant C\big(N_\varepsilon(T) + \delta\big)\left\|\psi(t)\right\|^2 + C\left\|\eta(t)\right\|^2, \tag{3.22}$$

where C *is a positive constant independent of* T *and* ε.

Lemma 3.7. *Suppose the same assumptions as in Proposition* 3.5 *hold. Then there exists a positive constant* δ_0 *such that if* $N_\varepsilon(T) + \delta + \varepsilon \leqslant \delta_0$, *then the following estimate holds for* $t \in [0, T]$:

$$\begin{aligned}
&\left\|(\psi, \eta, \sigma_x, \varepsilon\psi_x)(t)\right\|^2 + \int_0^t \left\|(\psi, \eta, \sigma_x, \varepsilon\psi_x)(\tau)\right\|^2 d\tau \\
&\quad \leqslant C\left\|(\psi, \eta, \sigma_x, \varepsilon\psi_x)(0)\right\|^2 + C\big(N_\varepsilon(T) + \delta + \varepsilon\big)M^2(t),
\end{aligned} \tag{3.23}$$

where C *is a positive constant independent of* T *and* ε.

Proof. First, integrating (3.19) over $[0, t] \times \Omega$ and substituting the estimate (3.20) to handle the integration of R_2, we have

$$\int_0^1 \mathcal{E}(t, x)\,dx + \int_0^t \int_0^1 \frac{1}{\tilde{\omega}^2}\eta^2\,dx\,d\tau = \int_0^1 \mathcal{E}(0, x)\,dx + \int_0^t \int_0^1 R_2\,dx\,d\tau \tag{3.24a}$$

$$\leqslant \int_0^1 \mathcal{E}(0, x)\,dx + C\big(N_\varepsilon(T) + \delta + \varepsilon^{3/2}\big)M^2(t) \tag{3.24b}$$

since $\int_0^1 R_{1x}\,dx = 0$ owing to the boundary conditions (3.13)–(3.15).

Multiply (3.11b) by $-\sigma_x$, integrate the resultant equality over $[0, t] \times \Omega$, apply the integration by parts, and then use Eq. (3.11c) and the boundary conditions (3.13) and (3.14), to obtain that

$$\begin{aligned}
&\int_0^t \int_0^1 K\big(\log(\psi + \tilde{\omega})^2 - \log\tilde{\omega}^2\big)(\psi + 2\tilde{\omega})\psi + \sigma_x^2 - \varepsilon^2\left(\frac{(\psi + \tilde{\omega})_{xx}}{\psi + \tilde{\omega}} - \frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)(\psi + 2\tilde{\omega})\psi\,dx\,d\tau \\
&= \int_0^1 \left(\frac{\eta + \tilde{j}}{(\psi + \tilde{\omega})^2} - \frac{\tilde{j}}{\tilde{\omega}^2}\right)\sigma_x(t, x) - \left(\frac{\eta + \tilde{j}}{(\psi + \tilde{\omega})^2} - \frac{\tilde{j}}{\tilde{\omega}^2}\right)\sigma_x(0, x)\,dx \\
&\quad + \int_0^t \int_0^1 \frac{1}{2}\left\{\left(\frac{\eta + \tilde{j}}{(\psi + \tilde{\omega})^2}\right)^2 - \left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)^2\right\}_x \sigma_x + \left(\frac{\eta + \tilde{j}}{(\psi + \tilde{\omega})^2} - \frac{\tilde{j}}{\tilde{\omega}^2}\right)(\sigma_x - \sigma_{xt})\,dx\,d\tau
\end{aligned} \tag{3.25a}$$

$$\leqslant C\big(\big\|(\psi,\eta,\sigma_x)(0)\big\|^2+\big\|(\psi,\eta,\sigma_x)(t)\big\|^2\big)$$
$$+\int_0^t C\big\|\eta(\tau)\big\|^2+\frac{1}{2}\big\|\sigma_x(\tau)\big\|^2\,d\tau+C\big(N_\varepsilon(T)+\delta\big)M^2(t). \tag{3.25b}$$

In deriving the above inequality, we have also used the Schwarz and the Sobolev inequalities as well as (1.27), (2.20), (3.21) and (3.22). We estimate each term in the left-hand side of (3.25a). The first term is estimated by $c\|\psi(t)\|^2$ below. Applying the integration by parts with using (3.13), we rewrite the third term of the left-hand side of (3.25a) as

$$(\text{3rd term})=-\varepsilon^2\int_0^t\int_0^1\frac{\psi+2\tilde\omega}{\psi+\tilde\omega}\psi_{xx}\psi-\frac{\psi+2\tilde\omega}{\tilde\omega(\psi+\tilde\omega)}\tilde\omega_{xx}\psi^2\,dx\,d\tau$$
$$=\varepsilon^2\int_0^t\int_0^1\left(1+\frac{\tilde\omega}{\psi+\tilde\omega}\right)\psi_x^2+\left(\frac{\tilde\omega}{\psi+\tilde\omega}\right)_x\psi_x\psi+\frac{\psi+2\tilde\omega}{\tilde\omega(\psi+\tilde\omega)}\tilde\omega_{xx}\psi^2\,dx\,d\tau \tag{3.26a}$$
$$\geqslant\int_0^t c\varepsilon^2\big\|\psi_x(\tau)\big\|^2\,d\tau-C\varepsilon^{3/2}M^2(t), \tag{3.26b}$$

where we have also used the Schwarz inequality with (2.16) and Corollary 2.3. Here c is a positive constant. Substitute these inequalities in (3.25), multiply the resultant inequality by μ, where μ is a positive constant to be determined, add the resultant inequality to (3.24), and then take μ and $N_\varepsilon(T)+\delta+\varepsilon$ sufficiently small. These procedures yield the desired estimate (3.23). □

3.4. Higher order estimates

This subsection is devoted to the derivation of higher order estimates. In the following discussion, we need justification of formal computations by using the mollifier with respect to time variable t since the regularity of the solution (ψ,η) constructed in Corollary 3.4 is not sufficient. However we omit this discussion since it is a standard argument. Hereafter, we use notations

$$A_i^2(t):=\big\|(\psi,\eta)(t)\big\|^2+\sum_{k=0}^{i}\big\|\big(\partial_t^k\psi_t,\partial_t^k\psi_x,\varepsilon\partial_t^k\psi_{xx}\big)(t)\big\|^2\quad\text{for } i\geqslant 0,$$
$$A_{-1}^2(t):=\big\|(\psi,\eta)(t)\big\|^2.$$

Differentiate (1.12b) with respect to x and divide the resultant equality by ω. Then rewrite the resultant equality by using Eq. (1.12a). On the other hand, differentiate (1.21a) with respect to x and divide resultant equality by $\tilde\omega$. Then take a difference of the resulting two equalities and differentiate the result with respect to t (see [8] in details). These computations give the equation

$$
\begin{aligned}
&2\partial_t^i\psi_{tt} - 2K\partial_t^i\psi_{xx} + \varepsilon^2\partial_t^i\psi_{xxxx} + 2\partial_t^i\psi_t \\
&\quad = 2\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3}\partial_t^i\eta_{xx} - 2\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\partial_t^i\psi_{xx} + \varepsilon^2\frac{(i+1)\psi_{xx}+2\tilde{\omega}_{xx}}{\psi+\tilde{\omega}}\partial_t^i\psi_{xx} \\
&\qquad + \sum_{l=1}^{4}\partial_t^i F_l + G_i \quad \text{for } i=0,1,
\end{aligned}
$$

$$
\begin{aligned}
&F_1 := -2\frac{\eta+2\tilde{j}}{(\psi+\tilde{\omega})^4}\tilde{\omega}_{xx}\eta + 2\tilde{j}\tilde{\omega}_{xx}\frac{(\psi+\tilde{\omega})^4-\tilde{\omega}^4}{(\psi+\tilde{\omega})^4\tilde{\omega}^4}, \qquad F_2 := \left(\frac{4K}{\tilde{\omega}}\tilde{\omega}_x - 2\tilde{\phi}_x\right)\psi_x, \\
&F_3 := \frac{2\eta_x^2}{(\psi+\tilde{\omega})^3} - \frac{8(\eta+\tilde{j})(\psi+\tilde{\omega})_x}{\psi+\tilde{\omega}}\eta_x + 2K\frac{\psi_x^2}{\tilde{\omega}} + 6\frac{(\eta+\tilde{j})^2(\psi+2\tilde{\omega})_x}{(\psi+\tilde{\omega})^5}\psi_x - \frac{2\psi_t^2}{\psi+\tilde{\omega}}, \\
&F_4 := 6\frac{\eta+2\tilde{j}}{(\psi+\tilde{\omega})^5}\tilde{\omega}_x^2\eta - 2K\frac{(\psi+\tilde{\omega})_x^2}{(\psi+\tilde{\omega})\tilde{\omega}}\psi - \bigl\{(\psi+\tilde{\omega})(\psi+2\tilde{\omega}) + \bigl(\tilde{\omega}^2 - D\bigr)\bigr\}\psi \\
&\qquad - 6\tilde{j}^2\tilde{\omega}_x^2\frac{(\psi+\tilde{\omega})^5-\tilde{\omega}^5}{(\psi+\tilde{\omega})^5\tilde{\omega}^5} - 2(\psi+\tilde{\omega})_x\sigma_x - \frac{\varepsilon^2\tilde{\omega}_{xx}^2}{2\tilde{\omega}(\psi+\tilde{\omega})}\psi, \\
&G_0 := 0, \qquad G_1 := 2\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3}\right)_t\eta_{xx} - 2\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)_t\psi_{xx} - \varepsilon^2\frac{\psi_{xx}+2\tilde{\omega}_{xx}}{(\psi+\tilde{\omega})^2}\psi_t\psi_{xx}.
\end{aligned}
\tag{3.27}
$$

The L^2-norms of F_1–F_4 are estimated as

$$
\begin{aligned}
&\|F_1\| \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\|\tilde{\omega}_{xx}\|\bigl(|\eta|_0+|\psi|_0\bigr) \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl(\|\eta\|_1+\|\psi_x\|\bigr), \\
&\|F_2\| \leqslant C\bigl(\varepsilon^{1/2}+\delta\bigr)\|\psi_x\|, \qquad \|F_3\| \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl\|(\eta_x,\psi_x,\psi_t)\bigr\|, \\
&\|F_4\| \leqslant C\bigl\|(\eta,\psi)\bigr\|,
\end{aligned}
\tag{3.28}
$$

where C is a positive constant independent of T and ε. In deriving the estimate of $\|F_2\|$, we have used Eq. (1.21a) and the inequality

$$
\left|\frac{4K}{\tilde{\omega}}\tilde{\omega}_x - 2\tilde{\phi}_x\right| = \left|-\frac{2\tilde{j}^2\tilde{\omega}_x}{\tilde{\omega}^5} + 2\varepsilon^2\left(\frac{\tilde{\omega}_{xx}}{\tilde{\omega}}\right)_x - \frac{2\tilde{j}}{\tilde{\omega}^2}\right| \leqslant C\bigl(\varepsilon^{1/2}+\delta\bigr),
$$

which follows from (1.27), (2.16), (2.20) and Corollary 2.3. The other estimates in (3.28) are proved by (1.27), (2.16), (2.20), (3.21) and Corollary 2.3. Similarly, we have that

$$
\begin{aligned}
&\|F_{1t}\| \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl(\|\eta_t\|_1+\|\psi_{xt}\|\bigr), \qquad \|F_{2t}\| \leqslant C\bigl(\varepsilon^{1/2}+\delta\bigr)\|\psi_{xt}\|, \\
&\|F_{3t}\| \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl\|(\eta_x,\psi_x,\psi_t,\eta_{xt},\psi_{xt},\psi_{tt})\bigr\|, \\
&\|F_{4t}\| \leqslant C\|\psi_t\| + C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl\|(\eta_t,\psi_{xt})\bigr\|, \\
&\|G_1\| \leqslant C\bigl(N_\varepsilon(T)+\delta\bigr)\bigl\|(\eta_{xx},\psi_{xx})\bigr\|
\end{aligned}
\tag{3.29}
$$

owing to the estimate

$$\left|(\psi_t, \eta_t)(t)\right|_0 \leqslant CN_\varepsilon(T), \tag{3.30}$$

where C is a positive constant independent of T and ε. We see that the estimate (3.30) holds by applying the inequality (2.16) on Eqs. (3.11a) and (3.11b).

Differentiating (3.11a) with respect to x yields

$$\partial_t^i \eta_{xx} = -2(\psi + \tilde{\omega})\partial_t^i \psi_{xt} + H_i, \tag{3.31}$$

$$H_0 := -2(\psi + \tilde{\omega})_x \psi_t, \qquad H_1 := -4(\psi + \tilde{\omega})_x \psi_{xt} - 2\psi_t \psi_{xt}.$$

The estimates

$$\left\|\partial_t^i \eta_x(t)\right\| \leqslant CA_i(t), \qquad \left\|\left(\eta_{xx}, \varepsilon \partial_x^3 \eta\right)(t)\right\| \leqslant CA_1(t) \tag{3.32}$$

for $i = 0, 1$ easily follow from Eqs. (3.11a) and (3.31). Moreover, it holds that

$$M^2(t) \leqslant C \int_0^t A_0^2(\tau)\, d\tau, \qquad \left\|\omega(t)\right\|_4 \leqslant C\left\|\psi(t)\right\|_2, \tag{3.33}$$

owing to (3.11c) and (3.32).

Lemma 3.8. *Suppose the same assumptions as in Proposition* 3.5 *hold. Then the estimate*

$$cA_1(t) \leqslant n_\varepsilon(t) \leqslant CA_1(t) \tag{3.34}$$

holds, where the positive constants c and C are independent of T and ε.

Proof. Let $i = 0$ in (3.27) for the moment. Multiply (3.27) by ψ_{xx} and apply integration by parts with using the boundary condition (3.14). The result is

$$\begin{aligned} 2K\|\psi_{xx}\|^2 + \varepsilon^2\|\psi_{xxx}\|^2 = {} & \int_0^1 \left(2\psi_{tt} + 2\psi_t - 2\frac{\eta + \tilde{j}}{(\psi + \tilde{\omega})^3}\eta_{xx} + 2\frac{(\eta + \tilde{j})^2}{(\psi + \tilde{\omega})^4}\psi_{xx}\right)\psi_{xx}\, dx \\ & - \int_0^1 \left(\varepsilon^2 \frac{\psi_{xx} + 2\tilde{\omega}_{xx}}{\psi + \tilde{\omega}}\psi_{xx} + \sum_{l=1}^4 F_l\right)\psi_{xx}\, dx. \end{aligned} \tag{3.35}$$

Applying the Schwarz inequality to the right-hand side of (3.35) together with (1.27), (2.16), (2.20), (3.28), (3.32) and Corollary 2.3, we have the estimate

$$\left\|\left(\psi_{xx}, \varepsilon \partial_x^3 \psi\right)(t)\right\| \leqslant CA_1(t). \tag{3.36}$$

Solving (3.27) with respect to $\varepsilon^2\psi_{xxxx}$, taking the L^2-norm, and then using the estimates (3.28), (3.32) and (3.36), we have $\varepsilon^2\|\partial_x^4\psi(t)\| \leqslant CA_1(t)$. Similarly as above, the estimate $\|\psi_{tt}(t)\| \leqslant$

$C(\|(\eta,\psi)(t)\|_2+\varepsilon^2\|\partial_x^4\psi(t)\|)$ follows. Due to (3.11a) and (3.31), $\|\psi_t(t)\|_l \leqslant C(\|\eta(t)\|_{l+1}+\|\psi(t)\|_l)$ for $l=0,1,2$ also holds. Subtract (1.21a) from (1.12b) and estimate the L^2-norm of the resultant equality by using (3.28), (3.32), (3.36). Hence, we have

$$\bigl\|\eta_t(t)\bigr\| \leqslant CA_1(t). \tag{3.37}$$

Consequently, these estimates mean (3.34). □

We derive the higher order estimates to complete the a priori estimate (3.16).

Lemma 3.9. *Suppose the same assumptions as in Proposition* 3.5 *hold. Then there exists a positive constant ε_0 such that if $N_\varepsilon(T)+\delta+\varepsilon\leqslant\varepsilon_0$, then the following estimate holds for $t\in[0,T]$ and $i=0,1$:*

$$\begin{aligned}
&\bigl\|\bigl(\partial_t^i\psi_t,\partial_t^i\psi_x,\varepsilon\partial_t^i\psi_{xx}\bigr)(t)\bigr\|^2+\int_0^t\bigl\|\bigl(\partial_t^i\psi_t,\partial_t^i\psi_x,\varepsilon\partial_t^i\psi_{xx}\bigr)(\tau)\bigr\|^2\,d\tau\\
&\quad\leqslant C\left(A_i^2(0)+\int_0^t A_{i-1}^2(\tau)\,d\tau\right).
\end{aligned}\tag{3.38}$$

Proof. Multiply (3.27) by $\partial_t^i\psi$, integrate the resultant equality by parts over Ω and use boundary conditions $(\psi,\partial_t^i\psi_t,\partial_t^i\psi_{xx})(t,0)=(\psi,\partial_t^i\psi_t,\partial_t^i\psi_{xx})(t,1)=0$. These computations yield that

$$I_1^{(i)}(t)+\int_0^t\int_0^1 2K\bigl(\partial_t^i\psi_x\bigr)^2+\varepsilon^2\bigl(\partial_t^i\psi_{xx}\bigr)^2\,dx\,d\tau$$

$$=I_1^{(i)}(0)+\int_0^t J_1^{(i)}(\tau)\,d\tau+\int_0^t\int_0^1 2\bigl(\partial_t^i\psi_t\bigr)^2\,dx\,d\tau,$$

$$I_1^{(i)}(t):=\int_0^1 2\partial_t^i\psi_t\partial_t^i\psi+\bigl(\partial_t^i\psi\bigr)^2\,dx,$$

$$\begin{aligned}
J_1^{(i)}(t):=\int_0^1&-2\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3}\partial_t^i\psi\right)_x\partial_t^i\eta_x+2\left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\partial_t^i\psi\right)_x\partial_t^i\psi_x\,dx\\
&+\int_0^1\varepsilon^2\frac{(i+1)\psi_{xx}+2\tilde{\omega}_{xx}}{\psi+\tilde{\omega}}\partial_t^i\psi_{xx}\partial_t^i\psi+\left(\sum_{l=1}^4\partial_t^iF_l\right)\partial_t^i\psi+G_i\partial_t^i\psi\,dx.
\end{aligned}\tag{3.39}$$

By the Schwarz inequality, we have

$$\bigl|I_1^{(i)}(t)\bigr|\leqslant CA_i^2(t).\tag{3.40}$$

Due to (2.16), Corollary 2.3 and the Schwarz inequality, the third term in $J_1^{(i)}(t)$ is estimated as

$$\left|(\text{3rd term})\right| \leqslant \int_0^1 \frac{\varepsilon^2}{4}\left(\partial_t^i \psi_{xx}\right)^2 dx + C\left(N_\varepsilon(T)+\varepsilon\right)A_i^2(t).$$

The other terms in $J_1^{(i)}(t)$ are estimated by using (3.28), (3.29), (3.32), (3.36), (3.37) and $|\eta+\tilde{j}|_1 \leqslant C(N_\varepsilon(T)+\delta)$, which follows from (1.27) and (2.16). Hence we have

$$\left|J_1^{(i)}(t)\right| \leqslant CA_{i-1}^2(t) + \int_0^1 \frac{\varepsilon^2}{4}\left(\partial_t^i \psi_{xx}\right)^2 dx + C\left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)A_i^2(t). \tag{3.41}$$

Substituting the estimates (3.40) and (3.41) in (3.39) gives

$$\begin{aligned} &I_1^{(i)}(t) + \int_0^t\int_0^1 2K\left(\partial_t^i\psi_x\right)^2 + \frac{3\varepsilon^2}{4}\left(\partial_t^i\psi_{xx}\right)^2 dx\,d\tau - \int_0^t\int_0^1 2\left(\partial_t^i\eta_t\right)^2 dx\,d\tau \\ &\quad\leqslant C\left(A_i^2(0) + \int_0^t A_{i-1}^2(\tau)\,d\tau + \left(N_\varepsilon(T)+\delta+\varepsilon^{1/2}\right)\int_0^t A_i^2(\tau)\,d\tau\right). \end{aligned} \tag{3.42}$$

Next, multiply (3.27) by $\partial_t^i\psi_t$ and integrate the resultant equality over Ω. This gives

$$\begin{aligned} &\int_0^1 \left(2\partial_t^i\psi_{tt} - 2K\partial_t^i\psi_{xx} + \varepsilon^2\partial_t^i\psi_{xxxx} + 2\partial_t^i\psi_t\right)\partial_t^i\psi_t\,dx \\ &\quad= \int_0^1 2\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3}\partial_t^i\eta_{xx}\partial_t^i\psi_t\,dx - \int_0^1 2\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\partial_t^i\psi_{xx}\partial_t^i\psi_t\,dx \\ &\qquad+ \int_0^1 \varepsilon^2\frac{(i+1)\psi_{xx}+2\tilde{\omega}_{xx}}{\psi+\tilde{\omega}}\partial_t^i\psi_{xx}\partial_t^i\psi_t\,dx + \int_0^1\left(\sum_{l=1}^4\partial_t^i F_l\right)\partial_t^i\psi_t + G_i\partial_t^i\psi_t\,dx. \end{aligned} \tag{3.43}$$

By the integration by parts and the boundary condition

$$\left(\partial_t^i\psi_t, \partial_t^i\psi_{xx}\right)(t,0) = \left(\partial_t^i\psi_t, \partial_t^i\psi_{xx}\right)(t,1) = 0,$$

we rewrite the left-hand side of the equality (3.43) as

$$(\text{L.H.S.}) = \frac{d}{dt}\int_0^1 \left(\partial_t^i\psi_t\right)^2 + K\left(\partial_t^i\psi_x\right)^2 + \frac{\varepsilon^2}{2}\left(\partial_t^i\psi_{xx}\right)^2 dx + \int_0^1 2\left(\partial_t^i\psi_t\right)^2 dx. \tag{3.44}$$

We rewrite the first term on the left-hand side of the equality (3.43), by substituting (3.31) in (3.43) and integrating by parts with aid of $\partial_t^i \psi_t(t,0) = \partial_t^i \psi_t(t,1) = 0$, to obtain

$$\begin{aligned}
(\text{1st term}) &= \int_0^1 -2\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3}\bigl(-2(\psi+\tilde{\omega})\partial_t^i \psi_{xt} + H_i\bigr)\partial_t^i \psi_t \, dx \\
&= -\int_0^1 2\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)_x \bigl(\partial_t^i \psi_t\bigr)^2 + 2\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3} H_i \partial_t^i \psi_t \, dx. \qquad (3.45)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\text{2nd term}) &= \int_0^1 2\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\partial_t^i \psi_x \partial_t^i \psi_{xt} + 2\left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\right)_x \partial_t^i \psi_x \partial_t^i \psi_t \, dx \\
&= \frac{d}{dt}\int_0^1 \frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\bigl(\partial_t^i \psi_x\bigr)^2 \, dx - \int_0^1 \left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\right)_t \bigl(\partial_t^i \psi_x\bigr)^2 \, dx \\
&\quad + \int_0^1 2\left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\right)_x \partial_t^i \psi_x \partial_t^i \psi_t \, dx. \qquad (3.46)
\end{aligned}$$

Substituting the equalities (3.44)–(3.46) in (3.43) and integrating the result over $(0,t)$ yield that

$$I_2^{(i)}(t) + \int_0^t \int_0^1 2\bigl(\partial_t^i \psi_t\bigr)^2 \, dx \, d\tau = I_2^{(i)}(0) + \int_0^t J_2^{(i)}(\tau)\, d\tau,$$

$$I_2^{(i)}(t) := \int_0^1 \bigl(\partial_t^i \psi_t\bigr)^2 + K\bigl(\partial_t^i \psi_x\bigr)^2 + \frac{\varepsilon^2}{2}\bigl(\partial_t^i \psi_{xx}\bigr)^2 - \frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\bigl(\partial_t^i \psi_x\bigr)^2 \, dx,$$

$$\begin{aligned}
J_2^{(i)}(t) &:= -\int_0^1 2\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)_x \bigl(\partial_t^i \psi_t\bigr)^2 + 2\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^3} H_i \partial_t^i \psi_t + \left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\right)_t \bigl(\partial_t^i \psi_x\bigr)^2 \, dx \\
&\quad + \int_0^1 2\left(\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\right)_x \partial_t^i \psi_x \partial_t^i \psi_t + \varepsilon^2 \frac{(i+1)\psi_{xx} + 2\tilde{\omega}_{xx}}{\psi+\tilde{\omega}} \partial_t^i \psi_{xx} \partial_t^i \psi_t \, dx \\
&\quad + \int_0^1 \left(\sum_{l=1}^4 \partial_t^i F_l\right)\partial_t^i \psi_t + G_i \partial_t^i \psi_t \, dx. \qquad (3.47)
\end{aligned}$$

The fourth term in $I_2^{(i)}(t)$ is estimated, with the aid of (1.27), (2.20) and the Sobolev inequality, as

$$\left|\int_0^1 -\frac{(\eta+\tilde{j})^2}{(\psi+\tilde{\omega})^4}\big(\partial_t^i\psi_x\big)^2\,dx\right| \leqslant C\big(N_\varepsilon(T)+\delta\big)\big\|\partial_t^i\psi_x(t)\big\|^2. \tag{3.48}$$

Moreover, we have

$$\left|\int_0^1 \partial_t^i F_4\partial_t^i\psi_t\,dx\right| \leqslant C_\nu A_{i-1}^2(t)+C\big(\nu+N_\varepsilon(T)+\delta\big)A_i(t)^2, \tag{3.49}$$

where the constant ν is positive and C_ν is a positive constant depending only on ν. The other terms in $I_2^{(i)}(t)$ and $J_2^{(i)}(t)$ are estimated similarly as the estimation of $I_1^{(i)}(t)$ and $J_1^{(i)}(t)$:

$$\big|I_2^{(i)}(t)\big| \leqslant CA_i^2(t), \tag{3.50}$$

$$\big|J_2^{(i)}(t)\big| \leqslant C_\nu A_{i-1}^2(t)+\int_0^1 \frac{\varepsilon^2}{4}\big(\partial_t^i\psi_{xx}\big)^2\,dx + C\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}+\nu\big)A_i^2(t), \tag{3.51}$$

where we have also used the estimate (3.30). Finally substituting (3.48)–(3.51) in (3.47) gives the inequality

$$\begin{aligned}
&\int_0^1 \big(\partial_t^i\psi_t\big)^2+K\big(\partial_t^i\psi_x\big)^2+\frac{\varepsilon^2}{2}\big(\partial_t^i\psi_{xx}\big)^2\,dx+\int_0^t\int_0^1 2\big(\partial_t^i\psi_t\big)^2\,dx\,d\tau\\
&\quad\leqslant CA_i^2(0)+C_\nu\int_0^t A_{i-1}^2(\tau)\,d\tau+C\big(N_\varepsilon(T)+\delta\big)\big\|\partial_t^i\psi_x(t)\big\|^2+\int_0^t\int_0^1 \frac{\varepsilon^2}{4}\big(\partial_t^i\psi_{xx}\big)^2\,dx\,d\tau\\
&\qquad+C\big(N_\varepsilon(T)+\delta+\varepsilon^{1/2}+\nu\big)\int_0^t A_i^2(\tau)\,d\tau.
\end{aligned} \tag{3.52}$$

Multiply (3.52) by 2, add the resulting inequality to (3.42) and then let both $N_\varepsilon(T)+\delta+\varepsilon^{1/2}$ and ν be sufficiently small. This computation gives the desired estimate (3.38). □

Proof of Proposition 3.5. Combining (3.23) with (3.38) and making $N_\varepsilon(T)+\delta+\varepsilon$ sufficiently small, we have the desired estimate (3.16). In this computation, we have also used the estimates (3.34) and (3.33). □

3.5. Decay estimate

Because the existence of the time global solution to the problem (1.1) and (1.4)–(1.7) is shown by the continuation argument on Corollary 3.4 and Proposition 3.5, it suffices to show the decay estimate (1.28) in order to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Substitute (3.26a) in (3.25a) and multiply the resultant equality by β, where β is a positive constant to be determined. Moreover, multiply (3.39) with $i=0$ by β^2, (3.47) with $i=0$ by $2\beta^2$, (3.39) with $i=1$ by β^3, (3.47) with $i=1$ by $2\beta^3$. Summing up (3.24a) and these results, we have

$$\begin{aligned}
&\hat{E}(t)+\int_0^t \hat{F}(\tau)\,d\tau=\hat{E}(0)\quad \text{for } t\in[0,\infty),\\
&\hat{E}(t):=\int_0^1 \mathcal{E}-\beta\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}-\frac{\tilde{j}}{\tilde{\omega}^2}\right)\sigma_x\,dx+\sum_{i=0}^{1}\beta^{i+2}\big(I_1^{(i)}+2I_2^{(i)}\big)(t),\\
&\hat{F}(t):=\int_0^1 \frac{\eta^2}{\tilde{\omega}^2}+\beta\left\{K\big(\log(\psi+\tilde{\omega})^2-\log\tilde{\omega}^2\big)(\psi+2\tilde{\omega})\psi+\sigma_x^2+\varepsilon^2\left(1+\frac{\tilde{\omega}}{\psi+\tilde{\omega}}\right)\psi_x^2\right\}dx\\
&\qquad+\sum_{i=0}^{1}\beta^{i+2}\int_0^1 2\big(\partial_t^i\psi_t\big)^2+2K\big(\partial_t^i\psi_x\big)^2+\varepsilon^2\big(\partial_t^i\psi_{xx}\big)^2\,dx-\int_0^1 R_2\,dx\\
&\qquad-\beta\int_0^1 \frac{1}{2}\left\{\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}\right)^2-\left(\frac{\tilde{j}}{\tilde{\omega}^2}\right)^2\right\}_x\sigma_x+\left(\frac{\eta+\tilde{j}}{(\psi+\tilde{\omega})^2}-\frac{\tilde{j}}{\tilde{\omega}^2}\right)(\sigma_x-\sigma_{xt})\,dx\\
&\qquad+\beta\int_0^1 \varepsilon^2\left\{\left(\frac{\tilde{\omega}}{\psi+\tilde{\omega}}\right)_x\psi_x\psi\,dx+\frac{\psi+2\tilde{\omega}}{\tilde{\omega}(\psi+\tilde{\omega})}\tilde{\omega}_{xx}\psi^2\right\}dx-\sum_{i=0}^{1}\beta^{i+2}\big(J_1^{(i)}+2J_2^{(i)}\big)(t).
\end{aligned}\tag{3.53}$$

Take β and $N_\varepsilon(T)+\delta+\varepsilon$ sufficiently small in this order so that $0<N_\varepsilon(T)+\delta+\varepsilon\ll\beta^3\ll\beta^2\ll\beta\ll1$. Then we see that both quantities $\hat{E}(t)$ and $\hat{F}(t)$ are equivalent to $A_1^2(t)$. Hence $\hat{E}(t)$ and $\hat{F}(t)$ are also equivalent to $n_\varepsilon^2(t)$ due to (3.34). In fact, we can confirm this assertion by applying the Schwarz inequality and the inequality (2.16) as well as the estimates (3.20), (3.28)–(3.30) and (3.32)–(3.37).

Since $\hat{E}(t)$ and $\hat{F}(t)$ are equivalent, there exists a certain positive constant α such that $\alpha\hat{E}(t)\leqslant\hat{F}(t)$. Differentiating (3.53) and substituting this inequality in the resultant equality yield the ordinary differential inequality

$$\frac{d}{dt}\hat{E}(t)+\alpha\hat{E}(t)\leqslant 0\quad \text{for } t\in[0,\infty).\tag{3.54}$$

As the quantity $\hat{E}(t)$ is also equivalent to $n_\varepsilon^2(t)$, solving (3.54) gives that

$$n_\varepsilon^2(t) \leqslant C n_\varepsilon^2(0) e^{-\alpha t}, \tag{3.55}$$

where C is positive constant independent of t and ε. The decay estimate (1.28) follows from the inequality (3.55) and the elliptic estimate (3.33). □

4. The classical limit

In this section, we discuss the classical limit from the quantum hydrodynamic model to the hydrodynamic model. Here and hereafter $(\rho^\varepsilon, j^\varepsilon, \phi^\varepsilon)$ denotes the solution to (1.1) and (1.4)–(1.7); $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon)$ denotes that to (1.17)–(1.20); (ρ^0, j^0, ϕ^0) denotes that to (1.4), (1.5), (1.7) and (1.29); $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\phi}^0)$ denotes that to (1.20), (1.18) and (1.30); $(\tilde{w}^\varepsilon, \tilde{j}^\varepsilon, \phi^\varepsilon)$ denotes that to (2.26)–(2.29).

Firstly, we study the classical limit for the stationary solution. Letting $\tilde{w}^0 := \log \rho^0$, we see that the solution $(\tilde{\omega}^0, \tilde{j}^0, \tilde{\phi}^0)$ satisfies

$$S\big[e^{\tilde{w}^0}, \tilde{j}^0\big]\tilde{w}_x^0 - \phi_x^0 = -\tilde{j}^0 e^{-\tilde{w}^0}, \tag{4.1a}$$

$$\tilde{\phi}_{xx}^0 = e^{\tilde{w}^0} - D. \tag{4.1b}$$

We introduce new functions as

$$\tilde{W}^\varepsilon := \tilde{w}^\varepsilon - \tilde{w}^0, \qquad \tilde{J}^\varepsilon := \tilde{j}^\varepsilon - \tilde{j}^0.$$

Proof of Lemma 1.6. First we show (1.33) by using Eqs. (2.26b) and (4.1a). Note that if δ is small enough, $\tilde{j}^0$ is written as $\tilde{j}^0 := \mathcal{J}[e^{\tilde{w}^0}]$ by the same method as the derivation of the formula (1.27). Hence the following estimate follows from the straight computation with the formula (1.27), the estimates (1.31) and (2.20):

$$\big|\tilde{J}^\varepsilon\big| \leqslant C|B_b|\big\|\tilde{W}^\varepsilon\big\| \leqslant C\delta\big\|\tilde{W}^\varepsilon\big\|. \tag{4.2}$$

Subtract (2.26b) from (4.1a), multiply the result by $\tilde{W}_x^\varepsilon$ and integrate the resultant equality over the domain Ω. Then integrate by parts and use $\tilde{W}^\varepsilon(0) = \tilde{W}^\varepsilon(1) = 0$, $(\tilde{w}_{xx}^\varepsilon + (\tilde{w}_x^\varepsilon)^2/2)(0) = (\tilde{w}_{xx}^\varepsilon + (\tilde{w}_x^\varepsilon)^2/2)(1) = 0$, (2.26c) and (4.1b). The result is

$$\begin{aligned}
&\int_0^1 S\big[e^{\tilde{w}}, \tilde{j}\big]\big(\tilde{W}_x^\varepsilon\big)^2 + \big(e^{\tilde{w}^\varepsilon} - e^{\tilde{w}^0}\big)\tilde{W}^\varepsilon\, dx \\
&\quad = \int_0^1 \left\{\left(\frac{\tilde{j}^\varepsilon}{e^{\tilde{w}^\varepsilon}} + \frac{\tilde{j}^0}{e^{\tilde{w}^0}}\right)\tilde{w}_x^\varepsilon - 1\right\}\left(\frac{\tilde{j}^\varepsilon}{e^{\tilde{w}^\varepsilon}} - \frac{\tilde{j}^0}{e^{\tilde{w}^0}}\right)\tilde{W}_x^\varepsilon + \frac{\varepsilon^2}{2}\left(\tilde{w}_{xx}^\varepsilon + \frac{(\tilde{w}_x^\varepsilon)^2}{2}\right)\tilde{W}_{xx}^\varepsilon\, dx.
\end{aligned} \tag{4.3}$$

The right-hand side of (4.3) is estimated by $C\delta\|\tilde{W}_x^\varepsilon\|^2 + C\varepsilon^2$ owing to the Hölder and the Poincaré inequalities, (1.31), (4.2) and Corollary 2.3. Notice that the second term on the left-hand side of the equality is positive. Since $K - (\tilde{j}^0 e^{-\tilde{w}^0})^2 - C\delta > 0$ holds if δ is sufficient small, we have $\|\tilde{W}_x^\varepsilon\| \leqslant C\varepsilon$. Also, by the Poincaré inequality, $\|\tilde{W}^\varepsilon\|_1 \leqslant C\varepsilon$ holds. Apparently, this estimate gives $\|(\tilde{\rho}^\varepsilon - \tilde{\rho}^0)\|_1 \leqslant C\varepsilon$. The other estimates of (1.33) are obtained by the estimate (4.2) and Eqs. (2.26c) and (4.1b).

Next we show (1.34). For this purpose, we show that $\|\tilde{W}_{xx}^\varepsilon\|$ converges to 0 as ε tends to 0, which gives the convergence of $\|(\partial_x^2\{\tilde{\rho}^\varepsilon - \tilde{\rho}^0\}, \partial_x^4\{\tilde{\phi}^\varepsilon - \tilde{\phi}^0\})\|$. We have from the boundedness of $\|\tilde{w}^\varepsilon\|_2$ and the convergence (1.33) that

$$\tilde{w}_{xx}^\varepsilon \rightharpoonup \tilde{w}_{xx}^0 \quad \text{in } L^2 \text{ weakly as } \varepsilon \to 0. \tag{4.4}$$

Differentiate Eq. (2.26b) and multiply the resultant equality by $\tilde{w}_{xx}^\varepsilon + (\tilde{w}_x^\varepsilon)^2/2$. Then integrating the resultant equality over the domain Ω by parts yields

$$\int_0^1 S\big[e^{\tilde{w}^\varepsilon}, \tilde{j}^\varepsilon\big]\big(\tilde{w}_{xx}^\varepsilon\big)^2 + \frac{\varepsilon^2}{2}\left\{\left(\tilde{w}_{xx}^\varepsilon + \frac{(\tilde{w}_x^\varepsilon)^2}{2}\right)_x\right\}^2 dx = Q\big[\tilde{w}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon\big],$$

$$\begin{aligned} Q\big[\tilde{w}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon\big] := -\int_0^1 & S\big[e^{\tilde{w}^\varepsilon}, \tilde{j}^\varepsilon\big]\frac{(\tilde{w}_x^\varepsilon)^2}{2}\tilde{w}_{xx}^\varepsilon \\ & + \left\{S\big[e^{\tilde{w}^\varepsilon}, \tilde{j}^\varepsilon\big]_x \tilde{w}_x^\varepsilon - \tilde{\phi}_{xx}^\varepsilon + \left(\frac{\tilde{j}^\varepsilon}{e^{\tilde{w}^\varepsilon}}\right)_x\right\}\left(\tilde{w}_{xx}^\varepsilon + \frac{(\tilde{w}_x^\varepsilon)^2}{2}\right) dx. \end{aligned} \tag{4.5}$$

Owing to (1.33), (4.4), Corollary 2.3 and the estimate $\|\tilde{W}^\varepsilon\|_1 \leqslant C\varepsilon$, the quantity $Q[\tilde{w}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\phi}^\varepsilon]$ converges to

$$Q\big[\tilde{w}^0, \tilde{j}^0, \tilde{\phi}^0\big] = \int_0^1 S\big[e^{\tilde{w}^0}, \tilde{j}^0\big]\big(\tilde{w}_{xx}^0\big)^2\, dx \tag{4.6}$$

as ε tends to 0. The equality (4.6) is shown by differentiating Eq. (4.1a) and multiplying the resultant equality by $\tilde{w}_{xx}^0 + (\tilde{w}_x^0)^2/2$. On the other hand, it holds that

$$\limsup_{\varepsilon\to 0} \int_0^1 S\big[e^{\tilde{w}^\varepsilon}, \tilde{j}^\varepsilon\big]\big(\tilde{w}_{xx}^\varepsilon\big)^2\, dx = \limsup_{\varepsilon\to 0} \int_0^1 S\big[e^{\tilde{w}^0}, \tilde{j}^0\big]\big(\tilde{w}_{xx}^\varepsilon\big)^2\, dx, \tag{4.7}$$

due to (1.33), Corollary 2.3 and $\|\tilde{W}^\varepsilon\|_1 \leqslant C\varepsilon$. Consequently, (4.5)–(4.7) yield the inequality

$$\limsup_{\varepsilon\to 0} \int_0^1 S\big[e^{\tilde{w}^0}, \tilde{j}^0\big]\big(\tilde{w}_{xx}^\varepsilon\big)^2\, dx \leqslant \int_0^1 S\big[e^{\tilde{w}^0}, \tilde{j}^0\big]\big(\tilde{w}_{xx}^0\big)^2\, dx. \tag{4.8}$$

Since $S[e^{\tilde{w}^0}, \tilde{j}^0] > c > 0$, we see from (4.4) and (4.8) that $\|\tilde{W}_{xx}^\varepsilon\|$ converges to 0.

We prove the convergence $\|(\varepsilon\partial_x^3\tilde{w}^\varepsilon, \varepsilon^2\partial_x^4\tilde{w}^\varepsilon)\| \to 0$, which immediately gives the convergence $\|(\varepsilon\partial_x^3\tilde{\rho}^\varepsilon, \varepsilon^2\partial_x^4\tilde{\rho}^\varepsilon)\| \to 0$ due to (2.20) and Corollary 2.3. By letting $\varepsilon \to 0$ in (4.5), we see

$\varepsilon\|(\tilde{w}^{\varepsilon}_{xx}+(\tilde{w}^{\varepsilon}_x)^2/2)_x\|\to 0$ holds. This together with Corollary 2.3 gives that $\varepsilon\|\partial_x^3\tilde{w}^{\varepsilon}\|\to 0$ as $\varepsilon\to 0$. Differentiating (2.26b) and taking the L^2-norm of the result, we obtain

$$\begin{aligned}\frac{\varepsilon^2}{2}\left\|\left(\tilde{w}^{\varepsilon}_{xx}+\frac{(\tilde{w}^{\varepsilon}_x)^2}{2}\right)_{xx}\right\| &= \hat{n}_1\big[\tilde{\omega}^{\varepsilon},\tilde{j}^{\varepsilon},\tilde{\phi}^{\varepsilon}\big]\\ &:= \big\|\big(S\big[e^{\tilde{w}^{\varepsilon}},\tilde{j}^{\varepsilon}\big]\tilde{w}^{\varepsilon}_x\big)_x-\tilde{\phi}^{\varepsilon}_{xx}+\big(\tilde{j}^{\varepsilon}e^{-\tilde{w}^{\varepsilon}}\big)_x\big\|. \end{aligned}\tag{4.9}$$

Note that $\varepsilon^2\|\{(\tilde{w}^{\varepsilon}_x)^2\}_{xx}\|$ converges to 0 as ε tends to 0, owing to Corollary 2.3. Moreover, $\hat{n}_1[\tilde{\omega}^{\varepsilon},\tilde{j}^{\varepsilon},\tilde{\phi}^{\varepsilon}]$ converges to $\hat{n}_1[\tilde{\omega}^{0},\tilde{j}^{0},\tilde{\phi}^{0}]$ as ε tends to 0. On the other hand, the equality $\hat{n}_1[\tilde{\omega}^{0},\tilde{j}^{0},\tilde{\phi}^{0}]=0$ follows from the differentiation of (4.1a) in terms of x. Consequently, we see that $\varepsilon^2\|\partial_x^4\tilde{w}^{\varepsilon}\|\to 0$ as $\varepsilon\to 0$. □

In order to study the classical limit for the non-stationary problem, we introduce new functions as

$$R^{\varepsilon}:=\rho^{\varepsilon}-\rho^0,\qquad J^{\varepsilon}:=j^{\varepsilon}-j^0,\qquad \Phi^{\varepsilon}:=\phi^{\varepsilon}-\phi^0.$$

Subtracting (1.1) from (1.29), we have the system of the equations

$$R^{\varepsilon}_t+J^{\varepsilon}_x=0,\tag{4.10a}$$

$$\begin{aligned}&J^{\varepsilon}_t+KR^{\varepsilon}_x-\left\{\left(\frac{j^{\varepsilon}}{\rho^{\varepsilon}}\right)^2\rho^{\varepsilon}_x-\left(\frac{j^0}{\rho^0}\right)^2\rho^0_x\right\}+\left(\frac{2j^{\varepsilon}}{\rho^{\varepsilon}}j^{\varepsilon}_x-\frac{2j^0}{\rho^0}j^0_x\right)\\ &\quad-\big(R^{\varepsilon}\phi^{\varepsilon}_x+\rho^0\Phi^{\varepsilon}_x\big)+J^{\varepsilon}=\varepsilon^2\rho^{\varepsilon}\left(\frac{(\sqrt{\rho^{\varepsilon}})_{xx}}{\sqrt{\rho^{\varepsilon}}}\right)_x,\end{aligned}\tag{4.10b}$$

$$\Phi^{\varepsilon}_{xx}=R^{\varepsilon}.\tag{4.10c}$$

The boundary condition is derived from (1.5) and (1.7) as

$$R^{\varepsilon}(t,0)=R^{\varepsilon}(t,1)=R^{\varepsilon}_t(t,0)=R^{\varepsilon}_t(t,1)=\Phi^{\varepsilon}(t,0)=\Phi^{\varepsilon}(t,1)=0.\tag{4.11}$$

Differentiating Eq. (4.10b) with respect to x and using Eq. (4.10a), we obtain the equation

$$\begin{aligned}&R^{\varepsilon}_{tt}-KR^{\varepsilon}_{xx}+\left\{\left(\frac{j^{\varepsilon}}{\rho^{\varepsilon}}\right)^2\rho^{\varepsilon}_x-\left(\frac{j^0}{\rho^0}\right)^2\rho^0_x\right\}_x-\left(\frac{2j^{\varepsilon}}{\rho^{\varepsilon}}j^{\varepsilon}_x-\frac{2j^0}{\rho^0}j^0_x\right)_x\\ &\quad+\big(R^{\varepsilon}\phi^{\varepsilon}_x+\rho^0\Phi^{\varepsilon}_x\big)_x+R^{\varepsilon}_t=-\varepsilon^2\left\{\rho^{\varepsilon}\left(\frac{(\sqrt{\rho^{\varepsilon}})_{xx}}{\sqrt{\rho^{\varepsilon}}}\right)_x\right\}_x.\end{aligned}\tag{4.12}$$

The following estimates had been obtained in [13]:

$$\big\|\big(\rho^0,j^0,\phi^0\big)(t)\big\|_2+\big\|\big(\rho^0_t,j^0_t\big)(t)\big\|_1\leqslant C,\tag{4.13a}$$

$$\rho^0,\ S\big[\rho^0,j^0\big]>c>0,\tag{4.13b}$$

where C and c are positive constants independent of t.

Proof of Theorem 1.7. Multiply Eq. (4.10b) by J^ε and integrate the resultant equality over the domain Ω. Then apply the integration by parts and use the boundary condition (1.6) to obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{2}(J^\varepsilon)^2\,dx + \int_0^1 (J^\varepsilon)^2\,dx \\
&\quad = \int_0^1 -KR_x^\varepsilon J^\varepsilon + \left\{\left(\frac{j^\varepsilon}{\rho^\varepsilon}\right)^2\rho_x^\varepsilon - \left(\frac{j^0}{\rho^0}\right)^2\rho_x^0\right\}J^\varepsilon\,dx \\
&\qquad + \int_0^1 -\left(\frac{2j^\varepsilon}{\rho^\varepsilon}j_x^\varepsilon - \frac{2j^0}{\rho^0}j_x^0\right)J^\varepsilon + \left(R^\varepsilon\phi_x^\varepsilon + \rho^0\Phi_x^\varepsilon\right)J^\varepsilon - \varepsilon^2\frac{(\sqrt{\rho^\varepsilon})_{xx}}{\sqrt{\rho^\varepsilon}}\left(\rho^\varepsilon J^\varepsilon\right)_x\,dx \\
&\quad \leqslant C\left\|\left(R^\varepsilon, R_x^\varepsilon, J^\varepsilon, J_x^\varepsilon\right)(t)\right\|^2 + C\varepsilon^2.
\end{aligned}
\tag{4.14}
$$

In deriving the inequality (4.14), we have estimated the last term on the right-hand side of this equality as

$$
\begin{aligned}
\left|\varepsilon^2\int_0^1 \frac{(\sqrt{\rho^\varepsilon})_{xx}}{\sqrt{\rho^\varepsilon}}\left(\rho^\varepsilon J^\varepsilon\right)_x\,dx\right| &\leqslant \varepsilon^2\left|\frac{1}{\sqrt{\rho^\varepsilon}}\right|_0 \left\|\left(\sqrt{\rho^\varepsilon}\right)_{xx}\right\|\left\{\left|\rho_x^\varepsilon\right|_0\left\|J^\varepsilon\right\| + \left|\rho^\varepsilon\right|_0\left\|J_x^\varepsilon\right\|\right\} \\
&\leqslant C\varepsilon^2,
\end{aligned}
\tag{4.15}
$$

and the other terms by the Schwarz inequality with (1.27), (2.20), (3.16), (4.13), Corollary 2.3 and $\|\Phi_x^\varepsilon(t)\|_1 \leqslant C\|R^\varepsilon(t)\|$.

Multiply Eq. (4.12) by R_t^ε and integrate the resultant equality over Ω. Then the integration by parts and the boundary condition (1.6) yield that

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \frac{1}{2}\left(R_t^\varepsilon\right)^2 + \frac{1}{2}S\left[\rho^0, j^0\right]\left(R_x^\varepsilon\right)^2\,dx + \int_0^1 \left(R_t^\varepsilon\right)^2\,dx = Q_3(t), \\
&Q_3(t) := -\int_0^1 \frac{1}{2}\left\{\left(\frac{j^0}{\rho^0}\right)^2\right\}_t\left(R_x^\varepsilon\right)^2 - \left(\left\{\left(\frac{j^0}{\rho^0}\right)^2 - \left(\frac{j^\varepsilon}{\rho^\varepsilon}\right)^2\right\}\rho_x^\varepsilon\right)_x R_t^\varepsilon - \left(\frac{j^0}{\rho^0}\right)_x\left(R_t^\varepsilon\right)^2\,dx \\
&\qquad + \int_0^1 \left\{\left(\frac{2j^\varepsilon}{\rho^\varepsilon} - \frac{2j^0}{\rho^0}\right)j_x^\varepsilon\right\}_x R_t^\varepsilon - \left(R^\varepsilon\phi_x^\varepsilon + \rho^0\Phi_x^\varepsilon\right)_x R_t^\varepsilon + \varepsilon^2\rho^\varepsilon\left(\frac{(\sqrt{\rho^\varepsilon})_{xx}}{\sqrt{\rho^\varepsilon}}\right)_x R_{xt}^\varepsilon\,dx.
\end{aligned}
\tag{4.16}
$$

The last term in $Q_3(t)$ is estimated by using (3.16), (3.34), (4.13) and Corollary 2.3 as

$$\begin{aligned}
\text{(last term)} &= \varepsilon^2 \int_0^1 \bigl\{\sqrt{\rho^\varepsilon}\bigl(\sqrt{\rho^\varepsilon}\bigr)_{xxx} - \bigl(\sqrt{\rho^\varepsilon}\bigr)_{xx}\bigl(\sqrt{\rho^\varepsilon}\bigr)_x\bigr\} R^\varepsilon_{xt}\,dx \\
&\leqslant \varepsilon^2 C\bigl(\bigl\|\bigl(\sqrt{\rho^\varepsilon}\bigr)_{xxx}(t)\bigr\| + \bigl\|\bigl(\sqrt{\rho^\varepsilon}\bigr)_{xx}(t)\bigr\|\bigr)\bigl\|R^\varepsilon_{xt}(t)\bigr\| \leqslant C\varepsilon. \qquad (4.17)
\end{aligned}$$

Substituting (4.17) in (4.16), and then applying the Sobolev and the Schwarz inequalities to the other terms in $Q_3(t)$ with using (3.16), (4.13), Corollary 2.3 and $\|\Phi^\varepsilon(t)\|_2 \leqslant C\|R^\varepsilon(t)\|$, we have

$$\frac{d}{dt}\int_0^1 \frac{1}{2}\bigl(R^\varepsilon_t\bigr)^2 + \frac{1}{2}S\bigl[\rho^0, j^0\bigr]\bigl(R^\varepsilon_x\bigr)^2\,dx + \int_0^1 \bigl(R^\varepsilon_t\bigr)^2\,dx \leqslant C\bigl\|\bigl(R^\varepsilon_t, R^\varepsilon_x, R^\varepsilon, J^\varepsilon_x, J^\varepsilon\bigr)(t)\bigr\|^2 + C\varepsilon. \tag{4.18}$$

Note that $R^\varepsilon_t(0,x) = J^\varepsilon_x(0,x) = 0$ and $\|R^\varepsilon_t(t)\| = \|J^\varepsilon_x(t)\|$ hold from (4.10a), and $\|R^\varepsilon(t)\| \leqslant C\|R^\varepsilon_x(t)\|$ holds from (2.17). Hence, by adding (4.14) to (4.18) and applying the Gronwall inequality to the resultant inequality, we get the desired estimate (1.36).

Finally we show the estimate (1.37). Let $\gamma \in (0, 1/2)$ be fixed, and $T_1 := (\log 1/\varepsilon^\gamma)/\beta$. For $t \leqslant T_1$, the estimate (1.36) yields that

$$\bigl\|\bigl(R^\varepsilon, J^\varepsilon\bigr)(t)\bigr\|_1 \leqslant \sqrt{\varepsilon}Ce^{\beta T_1} \leqslant C\varepsilon^{(1/2)-\gamma}. \tag{4.19}$$

For $T_1 \leqslant t$, using the estimates (1.28), (1.32) and (1.33), we obtain

$$\begin{aligned}
\bigl\|\bigl(R^\varepsilon, J^\varepsilon\bigr)(t)\bigr\|_1 &\leqslant C\bigl\|\bigl(\rho^\varepsilon - \tilde\rho^\varepsilon, j^\varepsilon - \tilde j^\varepsilon, \rho^0 - \tilde\rho^0, j^0 - \tilde j^0, \tilde\rho^\varepsilon - \tilde\rho^0, \tilde j^\varepsilon - \tilde j^0\bigr)(t)\bigr\|_1 \\
&\leqslant C\bigl(e^{-\alpha_1 T_1} + e^{-\alpha_2 T_1} + \varepsilon\bigr) \leqslant C\bigl(\varepsilon^{\alpha_1\gamma/\beta} + \varepsilon^{\alpha_2\gamma/\beta} + \varepsilon\bigr). \qquad (4.20)
\end{aligned}$$

Owing to (4.19) and (4.20), $\sup\|(R^\varepsilon, J^\varepsilon)(t)\|_1$ converges to 0 as ε tends to 0. The other assertion in Theorem 1.7 follows from the estimate $\|\Phi^\varepsilon(t)\|_3 \leqslant C\|R^\varepsilon(t)\|_1$. □

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Appendix A

This section is devoted to the discussion on the solvability of the linearized problem (1.9), (1.13)–(1.15) and (3.1). To this end, we study the general scalar equation

$$u_{tt} + L_1 u_t + b_1 u_t + b_2 u_x + b_3 u_{xx} + L_2 u = f, \tag{A.1a}$$

$$L_1 := b\partial_x, \quad b \in \mathcal{B}^1\bigl([0,1]\times[0,T]\bigr), \qquad L_2 := a\partial_x^4, \quad a > 0, \tag{A.1b}$$

$$b_1, b_2, b_3 \in \mathcal{B}^0\bigl([0,1]\times[0,T]\bigr) \cap C^1\bigl([0,T] : L^2\bigr), \qquad f \in C^1\bigl([0,T] : L^2\bigr) \tag{A.1c}$$

with initial and boundary data

$$u(0,x)=u_1(x)\in\mathcal{H},\qquad u_t(0,x)=u_2(x)\in H_0^1\cap H^2,\tag{A.2}$$

$$u(t,0)=u(t,1)=0,\qquad u_{xx}(t,0)=u_{xx}(t,1)=0,\tag{A.3}$$

where $\mathcal{H}:=\{g\in H_0^1\cap H^4;\ g_{xx}(0)=g_{xx}(1)=0\}$.

Lemma A.1. *The initial boundary value problem* (A.1)–(A.3) *has a unique solution* $u\in\overline{\mathfrak{X}}_4([0,T])$.

We construct a solution to the linearized problem (1.9), (1.13)–(1.15) and (3.1) by applying Lemma A.1, which is proved by the Galerkin method later (see [16] and [17]). Differentiating (3.1b) with respect to x, dividing the resultant equality by 2ω, using Eq. (3.1a), and then letting $U:=\hat{\omega}$, we have a scalar equation

$$U_{tt}+\bar{b}\partial_xU_t+\bar{b}_1U_t+\bar{b}_2U_x+\bar{b}_3U_{xx}+\bar{a}\partial_x^4U=\bar{f},$$

$$\bar{a}:=\frac{\varepsilon^2}{2},\qquad \bar{b}:=\frac{2j}{\omega^2},\qquad \bar{b}_1:=\frac{1}{\omega}\left\{\left(\frac{2j}{\omega}\right)_x+\omega_t\right\},\qquad \bar{b}_2:=-\frac{1}{\omega}\big(S\big[\omega^2,j\big]\omega\big)_x,$$

$$\bar{b}_3:=-S\big[\omega^2,j\big]-\frac{\varepsilon^2}{2}\frac{\omega_{xx}}{\omega},\qquad \bar{f}:=-\big(\omega^2\phi_x-j\big)_x\frac{1}{2\omega}\tag{A.4}$$

with the initial and the boundary conditions

$$U(0,x)=\omega_0,\qquad U_t(0,x)=\frac{-j_{0x}}{2\omega_0},\tag{A.5}$$

$$U(t,0)=\omega_l,\qquad U(t,1)=\omega_r,\qquad U_{xx}(t,0)=U_{xx}(t,1)=0.\tag{A.6}$$

Here the initial data (A.5) follows from (3.1a). Note that if $(\hat{\omega},\hat{j})\in\overline{\mathfrak{X}}_4([0,T])\times\overline{\mathfrak{X}}_3([0,T])$ is a solution to the problem (1.9), (1.13)–(1.15) and (3.1), then $U=\hat{\omega}\in\overline{\mathfrak{X}}_4([0,T])$ satisfies (A.4)–(A.6).

Lemma A.1 ensures the existence of solution U to (A.4)–(A.6) as follows. Define $\overline{U}:=U-A$, where $A(x)=\omega_l(1-x)+\omega_rx$, to obtain from (A.4)–(A.6) that

$$\overline{U}_{tt}+\bar{b}\partial_x\overline{U}_t+\bar{b}_1\overline{U}_t+\bar{b}_2\overline{U}_x+\bar{b}_3\overline{U}_{xx}+\bar{a}\partial_x^4\overline{U}=\bar{f}+\frac{1}{\omega}\big(S\big[\omega^2,j\big]\omega\big)_xA_x,\tag{A.7}$$

$$\overline{U}(0,x)=\omega_0-A,\qquad \overline{U}_t(0,x)=\frac{-j_{0x}}{2\omega_0},\tag{A.8}$$

$$\overline{U}(t,0)=\overline{U}(t,1)=0,\qquad \overline{U}_{xx}(t,0)=\overline{U}_{xx}(t,0)=0.\tag{A.9}$$

Notice that the coefficients and the left-hand side of (A.7) satisfy the conditions (A.1b) and (A.1c), since (ω,j) belongs to $\overline{\mathfrak{X}}_4([0,T])\times\overline{\mathfrak{X}}_3([0,T])$. In addition, the initial data (A.8) also verifies the condition (A.2) owing to the compatibility conditions (1.8). Hence we have a unique solution to the problem (A.4)–(A.6) due to Lemma A.1.

We proceed to construct the solution $(\hat{\omega},\hat{j})$ to the initial boundary value problem (1.9), (1.13)–(1.15) and (3.1) from U thus constructed. Define $(\hat{\omega},\hat{j})$ by

$$\hat{\omega}(t,x) := U(t,x), \tag{A.10a}$$

$$\hat{j}(t,x) := \int_0^x -2\omega U_t(t,x)\,dx + \hat{j}(t,0), \tag{A.10b}$$

$$\hat{j}(t,0) := \int_0^t \left\{ \frac{4j}{\omega} U_t - 2S\big[\omega^2, j\big]\omega U_x + \varepsilon^2 \omega^2 \left(\frac{U_{xx}}{\omega} \right)_x + \phi_x \omega^2 - j \right\}(t,0)\,dt + j_0(0).$$

It suffices to show that $(\hat{\omega}, \hat{j}) \in \overline{\mathfrak{X}}_4([0,T]) \times \overline{\mathfrak{X}}_3([0,T])$ is a desired solution to the linearized problem (1.9), (1.13)–(1.15) and (3.1). Apparently, the equality $\hat{j}_x = -2\omega\hat{\omega}_t$ holds from (A.10b). In addition, differentiating (A.10b) with respect to t and using (A.4), we have the equality

$$\begin{aligned}
\hat{j}_t(t,x) &= \int_0^x \left\{ \frac{4j}{\omega} U_t - 2S\big[\omega^2, j\big]\omega U_x + \varepsilon^2 \omega^2 \left(\frac{U_{xx}}{\omega} \right)_x + \omega^2 \phi_x - j \right\}_x (t,x)\,dx \\
&\quad + \left\{ \frac{4j}{\omega} U_t - 2S\big[\omega^2, j\big]\omega U_x + \varepsilon^2 \omega^2 \left(\frac{U_{xx}}{\omega} \right)_x + \omega^2 \phi_x - j \right\}(t,0) \\
&= \left\{ -\frac{2j}{\omega^2} \hat{j}_x - 2S\big[\omega^2, j\big]\omega\hat{\omega}_x + \varepsilon^2 \omega^2 \left(\frac{\hat{\omega}_{xx}}{\omega} \right)_x + \omega^2 \phi_x - j \right\}(t,x),
\end{aligned}$$

where we have also used $2\omega U_t = -\hat{j}_x$ and $U = \hat{\omega}$. Thus, $(\hat{\omega}, \hat{j})$ satisfies Eq. (3.1). Next, we confirm that $(\hat{\omega}, \hat{j})$ satisfies initial condition (1.4). Actually, the equalities $\hat{\omega}(0,x) = U(0,x) = \omega_0(x)$ and $\hat{j}(0,x) = \int_0^x j_{0x}\,dx + j_0(0) = j_0(x)$ hold from (A.10b) and (A.5). Moreover, the boundary conditions (1.14) and (1.15) immediately follow from (A.6). Consequently, $(\hat{\omega}, \hat{j})$ is the solution to the linearized problem (1.9), (1.13)–(1.15) and (3.1).

Proof of Lemma A.1. First, we consider the problem (A.1)–(A.3) for the initial data $u(0,x) = u_t(0,x) = 0$. Define the sequence $\{v_l(x) := \sqrt{2}\sin l\pi x\}_{l=1}^{\infty}$, which is a complete orthonormal system in L^2, and make an approximate sequence $\{u^n(t,x) := \sum_{l=1}^n a_l^n(t) v_l(x)\}_{n=1}^{\infty}$ by solving an ordinary differential equation for $a_l^n(t)$:

$$\big(u_{tt}^n, v_l\big) + \big(L_1 u_t^n, v_l\big) + \big(b_1 u_t^n, v_l\big) + \big(b_2 u_x^n + b_3 u_{xx}^n, v_l\big) + \big(L_2 u^n, v_l\big) = (f, v_l), \tag{A.11}$$

$$a_l^n(0) = a_{lt}^n(0) = 0 \tag{A.12}$$

for $l = 1, 2, \ldots, n$, where $(\cdot,\cdot)$ denotes a standard L^2-inner product. This ordinary differential equation has a unique solution $a_l^n \in \mathcal{B}^3([0,T])$ owing to the standard theory of the ordinary differential equations. Thus we see that u^n belongs to the function space $C^3([0,T]; \mathcal{H})$. Multiply (A.11) by a_{lt}^n and sum up the resultant equalities for $l = 1, 2, \ldots, n$ to obtain that

$$\big(u_{tt}^n, u_t^n\big) + \big(b u_{xt}^n, u_t^n\big) + \big(b_1 u_t^n, u_t^n\big) + \big(b_2 u_x^n + b_3 u_{xx}^n, u_t^n\big) + a\big(u_{xxxx}^n, u_t^n\big) = \big(f, u_t^n\big). \tag{A.13}$$

Integrating (A.13) by parts, with using the boundary condition $v_l(0) = v_l(1) = v_{lxx}(0) = v_{lxx}(1) = 0$ as well as (2.17), (2.18) and the Schwarz inequality, yields that

$$\frac{d}{dt}\big(\|u_t^n(t)\|^2 + \|u_{xx}^n(t)\|^2\big) \leqslant C\big\|\big(u_t^n, u_{xx}^n, f\big)(t)\big\|^2. \tag{A.14}$$

Differentiating Eq. (A.11) with respect to t, multiplying (A.11) by a_{ltt}^n and summing up the resultant equalities for $l = 1, 2, \ldots, n$, we have

$$\begin{aligned}&\big(u_{ttt}^n, u_{tt}^n\big) + \big(bu_{xtt}^n, u_{tt}^n\big) + \big(b_1u_{tt}^n + b_2u_{xt}^n + b_3u_{xxt}^n, u_{tt}^n\big) + a\big(u_{xxxxt}^n, u_{tt}^n\big)\\ &\quad + \big(b_tu_{xt}^n, u_{tt}^n\big) + \big(b_{1t}u_t^n + b_{2t}u_x^n + b_{3t}u_{xx}^n, u_{tt}^n\big) = \big(f_t, u_{tt}^n\big).\end{aligned} \tag{A.15}$$

Then the same argument as the derivation of (A.14) yields

$$\frac{d}{dt}\big(\|u_{tt}^n(t)\|^2 + \|u_{xxt}^n(t)\|^2\big) \leqslant C\big\|\big(u_{tt}^n, u_{xx}^n, u_{xxt}^n, u_{xxxx}^n, f_t\big)(t)\big\|^2 \tag{A.16}$$

owing to (2.16). Multiply (A.11) by $(l\pi)^4a_l^n$ for $l = 1, \ldots, n$, corresponding to $a_l^n\partial_x^4$, sum up the resultants for $l = 1, \ldots, n$ and then apply the Schwarz inequality to the resultant equality. The result is

$$\|u_{xxxx}^n(t)\|^2 \leqslant C\big\|\big(u_{tt}^n, u_{xxt}^n, u_{xx}^n\big)(t)\big\|^2, \tag{A.17}$$

where we have also used (2.17) and (2.18). On the other hand, since $\{v_l\}_{i=1}^{\infty}$ is a complete orthonormal system in L^2, substituting $t = 0$ in Eq. (A.1a) and using the Bessel inequality yield that

$$\|u_{tt}^n(0)\| \leqslant \|f(0)\|. \tag{A.18}$$

Add (A.14) to (A.16) and substitute (A.17) in the resultant inequality. Moreover apply the Gronwall inequality to the resultant inequality with $u^n(t) = 0$, $u_t^n(t) = 0$ and then substitute (A.18) to obtain

$$\big\|\big(u_t^n, u_{xx}^n, u_{tt}^n, u_{xxt}^n, u_{xxxx}^n\big)(t)\big\| \leqslant C, \tag{A.19}$$

where C is a constant depending on T but independent of t. Consequently, we see that $\{(u_t^n, u_{xx}^n, u_{tt}^n, u_{xxt}^n)\}_{n=1}^{\infty}$ is a bounded sequence in L^2. The inequalities (2.17), (2.18) and (A.19) show that the sequence $\{u^n\}_{n=1}^{\infty}$ is bounded in $C([0,T]; \mathcal{H}) \cap C^1([0,T]; H_0^1 \cap H^2) \cap C^2([0,T]; L^2)$. Hence there exist a subsequence, still denoted by $\{u^n\}_{n=1}^{\infty}$, and u such that

$$\begin{aligned}
u^n &\to u && \text{in } C\big([0,T]; H_0^1 \cap H^2\big) \cap C^1\big([0,T]; L^2\big) \text{ strongly},\\
u^n &\rightharpoonup u && \text{in } L^\infty(0,T; \mathcal{H}) \text{ weakly-star},\\
u_t^n &\rightharpoonup u_t && \text{in } L^\infty\big(0,T; H_0^1 \cap H^2\big) \text{ weakly-star},\\
u_{tt}^n &\rightharpoonup u_{tt} && \text{in } L^\infty\big(0,T; L^2\big) \text{ weakly-star},
\end{aligned}$$

as n tends to infinity since $\mathcal{H}$ and $H_0^1 \cap H^2$ are the Hilbert spaces. Passing to the limit in (A.11), we see that u is a solution to the problem (A.1a) and (A.3) with the initial data

$u(0,x)=u(1,x)=0$ in distribution sense. In fact, the boundary condition (A.3) holds as $u \in C([0,T]; H_0^1 \cap H^2) \cap L^\infty(0,T;\mathcal{H})$. By the standard theory (see [15] for example), we see that $(u_{tt}, u_{txx}, u_{xxxx})(t)$ is continuous in L^2 at $t=0$. The argument with using the mollifier with respect to time variable t gives that $u \in \overline{\mathfrak{X}}_4[0,T]$. Consequently, we complete the proof for the initial data $u(0,x)=u(1,x)=0$.

Finally we treat this initial boundary value problem (A.1)–(A.3) for the general initial data (A.2). We can pick up an approximation sequence $\{u_2^k\}_{k=0}^\infty \subset \mathcal{H}$ such that u_2^k converges to u_2 strongly in $H_0^1 \cap H^2$ as k tends to infinity since $\{v_l(x)/\sqrt{1+(l\pi)^2+(l\pi)^4}\}_{l=1}^\infty \subset \mathcal{H}$ is a complete orthonormal system in $H_0^1 \cap H^2$. We define a function u^k by solving the initial boundary value problem (A.1)–(A.3) with the initial data $u(0,x)=u_1(x)$ and $u_t(0,x)=u_2^k(x)$. For this purposes, let $\bar{u}^k := u^k - u_1 - u_2^k t$ and rewrite this problem to the one for $\bar{u}^k$ as

$$\begin{aligned} &\bar{u}^k_{tt} + L_1\bar{u}^k_t + b_1\bar{u}^k_t + b_2\bar{u}^k_x + b_3\bar{u}^k_{xx} + L_2\bar{u}^k \\ &\quad = f - bu^k_{2x} - b_1u^k_2 - b_2\big(u^k_2t+u_1\big)_x - b_3\partial_x^2\big(u^k_2t+u_1\big) - a\partial_x^4\big(u^k_2t+u_1\big), \end{aligned} \tag{A.20a}$$

$$\bar{u}^k(0,x) = \bar{u}^k_t(0,x) = 0, \tag{A.20b}$$

$$\bar{u}^k(t,0) = \bar{u}^k(t,1) = 0, \qquad \bar{u}^k_{xx}(t,0) = \bar{u}^k_{xx}(t,1) = 0. \tag{A.20c}$$

Note that the right-hand side of (A.20a) belongs to $C([0,T]:L^2)$ since $u_1, u_2^k \in \mathcal{H}$. Owing to the above discussion, the initial boundary value problem (A.20) has a solution $\bar{u}^k \in \overline{\mathfrak{X}}^4([0,T])$. As a consequence, we see that $u^k = \bar{u}^k + u_1 + u_2^k t$ is a solution to the initial boundary value problem (A.1)–(A.3) with $u(0,x)=u_1$ and $u_t(0,x)=u_2^k$. Applying the energy method on the equations for $u^k - u^l$ for $k,\, l = 0,1,2,\ldots,$ we also see that $\{u^k\}_{k=0}^\infty$ is the Cauchy sequence in $\overline{\mathfrak{X}}_4([0,T])$. Hence, there exists a certain function $u \in \overline{\mathfrak{X}}_4([0,T])$ such that $u^k \to u$ strongly in $\overline{\mathfrak{X}}_4([0,T])$ as $k \to \infty$. Apparently, the function u is a desired solution to the problem (A.1)–(A.3). Its uniqueness also follows from the standard energy method. Hence, the proof of Lemma A.1 is completed. □

References

[1] M.G. Ancona, G.J. Iafrate, Quantum correction to the equation of state of an electron gas in a semiconductor, Phys. Rev. B 39 (1989) 9536–9540.

[2] K. Bløtekjær, Transport equations for electrons in two-valley semiconductors, IEEE Trans. Electron Devices 17 (1970) 38–47.

[3] P. Degond, P. Markowich, On a one-dimensional steady-state hydrodynamic model, Appl. Math. Lett. 3 (1990) 25–29.

[4] C.L. Gardner, The quantum hydrodynamic model for semiconductor devices, SIAM J. Appl. Math. 54 (1994) 409–427.

[5] Y. Guo, W. Strauss, Stability of semiconductor states with insulating and contact boundary conditions, Arch. Ration. Mech. Anal. 179 (2006) 1–30.

[6] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1983.

[7] A. Jüngel, H. Li, On a one-dimensional steady-state hydrodynamic model, Arch. Math. (Brno) 40 (2004) 435–456.

[8] A. Jüngel, H. Li, Quantum Euler–Poisson systems: Global existence and exponential decay, Quart. Appl. Math. 62 (2004) 569–600.

[9] S. Kawashima, Y. Nikkuni, S. Nishibata, The initial value problem for hyperbolic–elliptic coupled systems and applications to radiation hydrodynamics, in: Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., vol. 99, 1999.

[10] S. Kawashima, Y. Nikkuni, S. Nishibata, Large-time behavior of solutions to hyperbolic–elliptic coupled systems, Arch. Ration. Mech. Anal. 170 (2003) 297–329.

[11] H. Li, P. Markowich, M. Mei, Asymptotic behavior of solutions of the hydrodynamic model of semiconductors, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002) 359–378.
[12] A. Matsumura, T. Murakami, Asymptotic behavior of solutions for a fluid dynamical model of semiconductor equation, Kyoto Univ. RIMS Kokyuroku 1495 (2006) 60–70.
[13] S. Nishibata, M. Suzuki, Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors, Osaka J. Math. 44 (2007) 639–665.
[14] R. Pinnau, A note on boundary conditions for quantum hydrodynamic equations, Appl. Math. Lett. 12 (1999) 77–82.
[15] R. Racke, Lectures on Nonlinear Evolution Equations. Initial Value Problems, Aspects Math., vol. E19, Friedr. Vieweg & Sohn, Braunschweig, 1992.
[16] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Appl. Math. Sci., vol. 68, Springer-Verlag, New York, 1988.
[17] E. Zeidler, Nonlinear Functional Analysis and Its Applications. II/A. Linear Monotone Operators; translated from the German by the author and Leo F. Boron, Springer-Verlag, New York, 1990.